

# KOORNWINDER POLYNOMIALS AND AFFINE HECKE ALGEBRAS

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**ABSTRACT.** In this paper we derive the bi-orthogonality relations, diagonal term evaluations and evaluation formulas for the non-symmetric Koornwinder polynomials. For the derivation we use certain representations of the (double) affine Hecke algebra which were originally defined by Noumi and Sahi. The structure of the diagonal terms is clarified by expressing them as residues of the bi-orthogonality weight function. We furthermore give the explicit connection between the non-symmetric and the (anti-)symmetric theory.

## 1. INTRODUCTION

Cherednik [2]–[6] and Macdonald [18] clarified the structure of Macdonald polynomials using certain representations of affine Hecke algebras in terms of difference-reflection operators. The underlying data stem from a fixed, reduced, irreducible root system  $\Sigma$ . The degrees of freedom are, besides the deformation parameter  $q$ , the number of different root length occurring in  $\Sigma$  (so at most two). In fact, their work shows that the Macdonald polynomials are naturally attached to the reduced, affine root system  $\tilde{\Sigma}$  associated with  $\Sigma$ .

As announced by Macdonald [4, sect. 8], and partially carried out by Noumi [21], Sahi [24], [25], Noumi & Stokman [22] and Nishino et al [20], the theory naturally extends to the setting of arbitrary (not necessarily reduced) irreducible affine root systems. This in particular allows to incorporate the very general six parameter family of Koornwinder [14] polynomials in the theory.

The affine root system underlying the Koornwinder polynomials is the non-reduced, irreducible affine root system  $S$  of type  $C^\vee C_n$ , which was introduced by Macdonald in [16]. The affine root system  $S$  contains all (possibly non-reduced) irreducible affine root systems of classical type as an affine root sub-system. On the polynomial level, this property is reflected by van Diejen's [8] observation that the families of Macdonald polynomials associated with classical root systems are special cases or limit cases of the Koornwinder polynomials. In fact, there is a large class of interesting families of multivariable orthogonal polynomials associated with classical root systems which are degenerate cases of the Koornwinder polynomials, see e.g. [27] and [10]. This supports the idea that the role of the Koornwinder polynomials in the theory of multivariable orthogonal polynomials associated with classical root systems is similar to the important and dominant role of the Askey-Wilson

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[1] polynomials in the theory of one variable (basic) hypergeometric orthogonal polynomials.

In this paper we continue the affine Hecke algebra approach to the theory of Koornwinder polynomials. In particular, Sahi's [25] bi-orthogonality relations for the non-symmetric Koornwinder polynomials are extended to the case of continuous parameter values, and the corresponding diagonal terms are evaluated explicitly. We furthermore derive the evaluation formulas for the non-symmetric Koornwinder polynomials. Anti-symmetric Koornwinder polynomials are defined, and the explicit connection between the non-symmetric and the (anti-)symmetric theory is established. This leads to new derivations of Koornwinder's [14], van Diejen's [9] and Sahi's [24] results on the orthogonality relations, quadratic norm evaluations and evaluation formulas for the symmetric Koornwinder polynomials. We also shortly discuss the analogue of Weyl's character formula for Koornwinder polynomials, and the (closely related) shift operators.

Instead of using shift operators to evaluate the diagonal terms for the non-symmetric Koornwinder polynomials, we use a method which is motivated by Cherednik's [5] beautiful approach for proving Opdam's [23] inversion formula of the non-symmetric Harish-Chandra transform (the so-called Cherednik-Opdam transform). This method amounts to evaluating the diagonal terms using an explicit description of the action of the double affine Hecke algebra on non-symmetric Koornwinder polynomials in terms of operators acting on the spectral parameter. This approach reveals interesting new structures, such as an expression of the diagonal terms as residues of the bi-orthogonality weight function. This method is also expected to be an important tool for obtaining a better understanding of  $q$ -analogues of the Cherednik-Opdam transform, see e.g. [7] and [13] for some preliminary considerations in the rank one setting.

The results of this paper extend the results of Macdonald [18] and Cherednik [4] on the bi-orthogonality relations, diagonal term evaluations and evaluation formulas for non-symmetric Macdonald polynomials associated with classical reduced root systems, as well as the results of Noumi & Stokman [22], in which the rank one setting was treated in detail.

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## 2. THE AFFINE ROOT SYSTEM OF TYPE $C^\vee C_n$

In this section we discuss the affine root system of type  $C^\vee C_n$ , which was introduced by Macdonald in [16]. Let  $n$  be a positive integer  $\geq 2$ . Let  $V = (\mathbb{R}^n, \langle \cdot, \cdot \rangle)$  be Euclidean  $n$ -space with orthonormal basis  $\{\epsilon_i\}_{i=1}^n$ . We write  $\widehat{V}$  for the affine linear transformations from  $V$  to  $\mathbb{R}$ . As a vector space,  $\widehat{V}$  can be identified with  $V \oplus \mathbb{R}\delta$ , where vectors in  $V$  are considered as linear functionals on  $V$  via the scalar product  $\langle \cdot, \cdot \rangle$ , and where  $\delta$  is the function identically equal to one on  $V$ . We extend

the scalar product  $\langle \cdot, \cdot \rangle$  to a positive semi-definite form on  $\widehat{V}$  by requiring that the constant function  $\delta$  is in the radical of  $\langle \cdot, \cdot \rangle$ .

Let  $S \subset \widehat{V}$  be the subset

$$\begin{aligned} S = & \{ \pm \epsilon_i + \frac{m}{2} \delta, \pm 2\epsilon_i + m\delta \mid m \in \mathbb{Z}, i = 1, \dots, n \} \\ & \cup \{ \pm \epsilon_i \pm \epsilon_j + m\delta \mid m \in \mathbb{Z}, 1 \leq i < j \leq n \}, \end{aligned} \quad (2.1)$$

where all the sign combinations occur. Let  $\mathcal{W} = \mathcal{W}(S)$  be the sub-group of  $\mathrm{GL}_{\mathbb{R}}(\widehat{V})$  generated by the reflections  $s_{\beta}$  ( $\beta \in S$ ), where

$$s_f(g) = g - \langle g, f^\vee \rangle f, \quad f \in \widehat{V} \setminus \mathbb{R}\delta, \quad g \in \widehat{V},$$

and where  $f^\vee = 2f/\langle f, f \rangle$  is the co-root of  $f$ . Observe that  $s_f(g) = g \circ \widetilde{s}_f^{-1}$ , with  $\widetilde{s}_f : V \rightarrow V$  the orthogonal reflection in the affine hyperplane  $f^{-1}(0)$ .

By [16],  $S \subset \widehat{V}$  is an irreducible, affine root system. In particular,  $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$  for all  $\alpha, \beta \in S$ , and  $S$  is stable under the action of  $\mathcal{W}$ . The sub-group  $\mathcal{W} \subset \mathrm{GL}(\widehat{V})$  is called the affine Weyl group of  $S$ .

Let  $R$  be the inmultipliable roots in  $S$  and  $R^\vee \subset S$  the corresponding co-root system. Then  $R$  and  $R^\vee$  are irreducible, reduced affine root systems in  $\widehat{V}$ , with affine Weyl group  $\mathcal{W}$ . The projection  $\Sigma \subset R$  of  $R$  on  $V$  along the direct sum decomposition  $\widehat{V} = V \oplus \mathbb{R}\delta$  is an irreducible root system of type  $C_n$  with Weyl group  $W = S_n \ltimes (\pm 1)^n \subset \mathcal{W}$  given by permutations and sign changes of the fixed basis  $\{\epsilon_i\}_{i=1}^n$  of  $V$  (here  $S_n$  denotes the symmetric group in  $n$  letters). Due to the (non-disjoint) union  $S = R \cup R^\vee$  of  $S$  into the reduced affine root sub-system  $R$  of type  $\widetilde{C}_n$  and its co-root system, we call  $S$  of type  $C^\vee C_n$ , cf. [16].

Let  $Q^\vee$  be the co-root lattice of  $\Sigma$ , which coincides with the weight lattice  $\Lambda$  of  $\Sigma$ . In fact,  $Q^\vee = \Lambda$  is the full  $\mathbb{Z}$ -lattice in  $V$  with basis  $\{\epsilon_i\}_{i=1}^n$ . Then

$$\mathcal{W} = W \ltimes \tau(Q^\vee),$$

where  $\tau(v) \in \mathrm{GL}(\widehat{V})$  ( $v \in V$ ) is the translation operator defined by  $\tau(v)f = f + \langle v, f \rangle \delta$  for  $f \in \widehat{V}$ . Observe that  $\tau(v)f = f \circ \widetilde{\tau}_v^{-1}$  with  $\widetilde{\tau}_v : V \rightarrow V$  given by  $\widetilde{\tau}_v(\lambda) = \lambda - v$ .

*Remark 2.1.* In Macdonald's [18] and Cherednik's [2]–[6] work the translation operator  $\tau(v)$  in fact corresponds to  $\tau(-v) = \tau(v)^{-1}$ . Later on, this change of convention (which is related to conjugation with the largest Weyl group element  $\sigma \in W$ ) causes certain changes of signs compared with Cherednik's and Macdonald's theory, see e.g. remark 4.10(ii).

We fix a basis  $\{a_i\}_{i=0}^n$  of  $R$  by

$$a_0 = \delta - 2\epsilon_1, \quad a_i = \epsilon_i - \epsilon_{i+1} \quad (i = 1, \dots, n-1), \quad a_n = 2\epsilon_n.$$

Observe that  $\{a_0^\vee = a_0/2, a_1, \dots, a_{n-1}, a_n^\vee = a_n/2\}$  is a basis of  $R^\vee$ , as well as of  $S$ . Furthermore,  $\{a_i\}_{i=1}^n$  is a basis of the gradient root system  $\Sigma$ . We write  $\Sigma^+$  (respectively  $\Sigma^-$ ) for the corresponding positive (respectively negative) roots in  $\Sigma$ , and  $\Lambda^+ = \bigoplus_{i=1}^n \mathbb{Z}_+ \omega_i$  for the corresponding cone of dominant weights of  $\Sigma$ . Here  $\omega_i = \epsilon_1 + \dots + \epsilon_i$  ( $i = 1, \dots, n$ ) are the fundamental weights of  $\Lambda$ , i.e.  $\langle \omega_i, a_j^\vee \rangle = \delta_{i,j}$  for all  $i, j = 1, \dots, n$ , where  $\delta_{i,j}$  is the Kronecker delta. We furthermore write  $Q^{\vee,+}$  for the positive span of the simple co-roots  $a_i^\vee$  ( $i = 1, \dots, n$ ).

Let  $R^+$  (respectively  $R^-$ ) be the positive (respectively negative) roots of  $R$  with respect to the basis of the previous paragraph. In particular,  $R^+ = \Sigma^+ \cup \{\beta \in R \mid \beta(0) > 0\}$ .

The affine Weyl group  $\mathcal{W}$  is generated by the simple reflections  $s_i = s_{a_i}$  ( $i = 0, \dots, n$ ). In fact,  $\mathcal{W}$  is isomorphic to the Coxeter group with generators  $s_i$  ( $i = 0, \dots, n$ ) satisfying  $s_i^2 = 1$  and the braid relations  $s_i s_{i+1} s_i s_{i+1} = s_{i+1} s_i s_{i+1} s_i$  ( $i = 0, i = n-1$ ),  $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$  ( $i = 1, \dots, n-2$ ) and  $s_i s_j = s_j s_i$  for  $|i-j| \geq 2$ .

With our present conventions, Lusztig's formula [15, 1.4(a)] for the length of an element in  $\mathcal{W}$  is given by

$$l(\tau(\lambda)w) = \sum_{\alpha \in \Sigma^+} |-\langle \lambda, w\alpha \rangle + \chi(w\alpha)|, \quad \lambda \in \Lambda, w \in W, \quad (2.2)$$

where  $\chi(\alpha) = 1$  if  $\alpha \in \Sigma^-$  and  $= 0$  otherwise.

We write  $\Sigma = \Sigma_m \cup \Sigma_l$  for the decomposition of  $\Sigma$  into  $W$ -orbits, where  $\Sigma_m$  (respectively  $\Sigma_l$ ) is the set of roots of length two (respectively four). We furthermore set  $\Sigma_s = \frac{1}{2}\Sigma_l$ . There are five  $\mathcal{W}$ -orbits in  $S$ , namely

$$\begin{aligned} \mathcal{W}a_0^\vee &= \left(\frac{1}{2} + \mathbb{Z}\right)\delta + \Sigma_s, & \mathcal{W}a_0 &= (1 + 2\mathbb{Z})\delta + \Sigma_l, \\ \mathcal{W}a_i &= \mathbb{Z}\delta + \Sigma_m & (i \in \{1, \dots, n-1\} \text{ arbitrary}), \\ \mathcal{W}a_n^\vee &= \mathbb{Z}\delta + \Sigma_s, & \mathcal{W}a_n &= 2\mathbb{Z}\delta + \Sigma_l. \end{aligned} \quad (2.3)$$

Observe that  $R$  (respectively  $R^\vee$ ) has three  $\mathcal{W}$ -orbits, namely  $\mathcal{W}a_0$ ,  $\mathcal{W}a_i$  and  $\mathcal{W}a_n$  (respectively  $\mathcal{W}a_0^\vee$ ,  $\mathcal{W}a_i$  and  $\mathcal{W}a_n^\vee$ ), where  $i \in \{1, \dots, n-1\}$  is arbitrary.

For later purposes, we define an action of  $\mathcal{W}$  on  $V$  which extends the canonical  $W$ -action on  $V$ . It suffices to specify the action of the simple reflection  $s_0$  on  $V$ , which we take to be

$$s_0.x = (-1 - x_1, x_2, \dots, x_n),$$

where  $x_i = \langle x, \epsilon_i \rangle$ . Observe that  $\Lambda \subset V$  is  $\mathcal{W}$ -stable, and that  $\tau(\lambda).x = x + \lambda$  for  $\lambda \in \Lambda$ . We denote this action of  $\mathcal{W}$  on  $V$  with a dot and we call it the *dot-action*, in order to avoid confusion with the canonical action of  $\mathcal{W}$  on  $\widehat{V}$  and its induced dual action on  $V$ .

### 3. THE (DOUBLE) AFFINE HECKE ALGEBRA

Let  $\mathcal{A}$  be the group algebra of the weight lattice  $\Lambda$ . We write  $x^\lambda$  ( $\lambda \in \Lambda$ ) for the canonical basis of  $\mathcal{A}$ , so that  $x^0 = 1$  is the unit element in  $\mathcal{A}$  and  $x^\lambda x^\mu = x^{\lambda+\mu}$  for all  $\lambda, \mu \in \Lambda$ . The group algebra  $\mathcal{A}$  is isomorphic to the Laurent polynomials in the  $n$  independent indeterminates  $x_i = x^{\epsilon_i}$  ( $i = 1, \dots, n$ ). Let  $q \in \mathbb{C} \setminus \{0\}$  be generic complex (in particular, not a root of unity) and let  $q^{1/2}$  be a fixed square root of  $q$ . We write  $x^{\mu+c\delta} = q^c x^\mu$  for  $\mu \in \Lambda$  and  $c \in \frac{1}{2}\mathbb{Z}$ . Then the assignment  $w(x^\mu) = x^{w\mu}$  for  $w \in \mathcal{W}$  and  $\mu \in \Lambda \subset \widehat{V}$  extends by linearity to an action of  $\mathcal{W}$  on  $\mathcal{A}$ . In particular, the action of the simple reflections  $s_i$  ( $i = 0, \dots, n$ ) on  $\mathcal{A}$  is given by

$$\begin{aligned} (s_0 f)(x) &= f(qx_1^{-1}, x_2, \dots, x_n), \\ (s_i f)(x) &= f(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n) & (i = 1, \dots, n-1), \\ (s_n f)(x) &= f(x_1, \dots, x_{n-1}, x_n^{-1}), \end{aligned} \quad (3.1)$$

where  $f \in \mathcal{A}$  and  $x = (x_1, \dots, x_n)$ . In particular, the translation operators  $\tau(\mu)$  ( $\mu \in \Lambda$ ) act as  $q$ -difference operators:  $\tau(\mu)(x^\lambda) = q^{\langle \mu, \lambda \rangle} x^\lambda$  for all  $\lambda, \mu \in \Lambda$ .

The Noumi [21] representation is a five (=  $\#\{\mathcal{W} - \text{orbits in } S\}$ ) parameter deformation of the above action of  $\mathcal{W}$  on the group algebra  $\mathcal{A}$ . We incorporate the five extra degrees of freedom in a so-called multiplicity function  $\mathbf{t} = (t_\beta)_{\beta \in S}$  of  $S$ , which is a  $\mathcal{W}$ -invariant map from  $S$  to  $\mathbb{C} \setminus \{0\}$  (so  $t_{w\beta} = t_\beta$  for all  $w \in \mathcal{W}$  and all  $\beta \in S$ ). We furthermore set  $t_f = 1$  if  $f \in \widehat{V} \setminus S$ . A multiplicity function is thus uniquely determined by the five values  $t_{a_0^\vee}, t_{a_0}, t_{a_i}$  ( $i \in \{1, \dots, n-1\}$  arbitrary),  $t_{a_n}$  and  $t_{a_n^\vee}$ . In order to avoid cumbersome notations, we sometimes write  $t_i$  (respectively  $t_i^\vee$ ) for  $t_{a_i}$  (respectively  $t_{a_i^\vee}$ ) and we write  $t$  for the value of  $t_j = t_j^\vee$  with  $j \in \{1, \dots, n-1\}$ . Furthermore, we use the short-hand notation  $\mathbf{k} = (t_\beta)_{\beta \in R} \simeq (t_0, t, t_n)$  and  $\mathbf{k}^\vee = (t_\beta)_{\beta \in R^\vee} \simeq (t_0^\vee, t, t_n^\vee)$  for the corresponding multiplicity functions of  $R$  and  $R^\vee$ , respectively. We assume throughout the paper that the values of the multiplicity function  $\mathbf{t}$  are generically complex.

In the Noumi representation, the role of the affine Weyl group  $\mathcal{W}$  is replaced by the affine Hecke algebra of type  $\tilde{C}_n$ , which is defined as follows.

**Definition 3.1.** *The affine Hecke algebra  $H = H(R; \mathbf{k})$  of type  $\tilde{C}_n$  is the unital, associative algebra with generators  $T_0, \dots, T_n$  and relations*

$$(T_i - t_i)(T_i + t_i^{-1}) = 0, \quad (i = 0, \dots, n), \quad (3.2)$$

and the braid relations

$$\begin{aligned} T_i T_{i+1} T_i T_{i+1} &= T_{i+1} T_i T_{i+1} T_i, & (i = 0, i = n-1), \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, & (i = 1, \dots, n-2), \\ T_i T_j &= T_j T_i, & |i - j| \geq 2. \end{aligned} \quad (3.3)$$

We call (3.2) and (3.3) the  $H(R; \mathbf{k})$ -relations for the  $(n+1)$ -tuple  $(T_0, \dots, T_n)$ . Furthermore, we write  $H(R^\vee; \mathbf{k}^\vee)$  for the affine Hecke algebra  $H$  in which the parameter  $t_i$  is replaced by  $t_i^\vee$  for  $i = 0, \dots, n$ .

We recall here some of the basic properties of the affine Hecke algebra  $H$ , see Lusztig [15] for details and for a general discussion on affine Hecke algebras.

For a reduced expression  $w = s_{i_1} \cdots s_{i_r}$  of  $w \in \mathcal{W}$  we set  $T_w = T_{i_1} \cdots T_{i_r}$ . This is independent of the choice of reduced expression by the braid relations (3.3) for the  $T_i$ , and  $\{T_w\}_{w \in \mathcal{W}}$  is a linear basis of  $H$ .

For  $\lambda \in \Lambda^+$  we set  $Y^\lambda = T_{\tau(\lambda)}$ , and for  $\lambda = \mu - \nu \in \Lambda$  with  $\mu, \nu \in \Lambda^+$  we set  $Y^\lambda = Y^\mu (Y^\nu)^{-1}$ . The length identity (2.2) implies that the  $Y^\lambda$  ( $\lambda \in \Lambda$ ) are well-defined (i.e. independent of the choice of decomposition  $\lambda = \mu - \nu$ ). Furthermore, the sub-space  $\mathcal{A}_Y = \text{span}\{Y^\lambda \mid \lambda \in \Lambda\}$  is a commutative subalgebra of  $H$  isomorphic to  $\mathcal{A}$  (in particular,  $Y^0 = 1$  and  $Y^\lambda Y^\mu = Y^{\lambda+\mu}$  for all  $\lambda, \mu \in \Lambda$ ). We identify  $f(x) = \sum_\lambda c_\lambda x^\lambda \in \mathcal{A}$  with  $f(Y) = \sum_\lambda c_\lambda Y^\lambda \in \mathcal{A}_Y$  in the remainder of the paper. We write  $Y_i = Y^{\epsilon_i}$ , which corresponds with  $x_i$  under the identification of  $\mathcal{A}_Y$  with  $\mathcal{A}$ .

The Hecke algebra  $H_0 = H_0(\Sigma; t, t_n)$  of the finite Weyl group  $W$  can be identified with the subalgebra of  $H$  generated by  $T_i$  ( $i = 1, \dots, n$ ). Then  $\{T_w\}_{w \in W}$  is a linear basis of  $H_0$  and

$$H \simeq H_0 \otimes \mathcal{A}_Y \simeq \mathcal{A}_Y \otimes H_0$$

as vector spaces by multiplication. The commutation relations between  $T_i \in H_0$  ( $i = 1, \dots, n$ ) and  $f(Y) \in \mathcal{A}_Y$  are given by the formulas

$$\begin{aligned} T_i f(Y) - (s_i f)(Y) T_i &= (t - t^{-1}) \left( \frac{f(Y) - (s_i f)(Y)}{1 - Y^{-a_i}} \right), \\ T_n f(Y) - (s_n f)(Y) T_n &= ((t_n - t_n^{-1}) + (t_0 - t_0^{-1}) Y_n^{-1}) \left( \frac{f(Y) - (s_n f)(Y)}{1 - Y_n^{-2}} \right) \end{aligned} \quad (3.4)$$

for  $i = 1, \dots, n-1$ , see [15, prop. 3.6]. These commutation relations can be used to prove inductively that

$$Y_i = T_i \cdots T_{n-1} T_n T_{n-1} \cdots T_1 T_0 T_1^{-1} T_2^{-1} \cdots T_{i-1}^{-1}, \quad i \in \{1, \dots, n\} \quad (3.5)$$

in  $H$ , see Noumi [21] or Sahi [24, (11)]. The expression (3.5) is the analogue in  $H$  of the reduced expression

$$\tau(\epsilon_i) = s_i \cdots s_{n-1} s_n s_{n-1} \cdots s_1 s_0 s_1 \cdots s_{i-1}, \quad i \in \{1, \dots, n\} \quad (3.6)$$

in  $\mathcal{W}$ . We define a rational function  $v_\beta(x) = v_\beta(x; \mathbf{t}; q) \in \mathbb{C}(x) = \text{Quot}(\mathcal{A})$  by

$$v_\beta(x; \mathbf{t}; q) = \frac{(1 - t_\beta t_{\beta/2} x^{\beta/2})(1 + t_\beta t_{\beta/2}^{-1} x^{\beta/2})}{(1 - x^\beta)}, \quad \beta \in R. \quad (3.7)$$

Observe that for  $\beta \in R$  with  $\beta/2 \notin S$ , the expression (3.7) reduces to  $v_\beta(x) = (1 - t_\beta^2 x^\beta)/(1 - x^\beta)$  since  $t_{\beta/2} = 1$ . The following crucial theorem was proved by Noumi [21].

**Theorem 3.2** (The Noumi representation). *The assignment*

$$T_i \mapsto t_i + t_i^{-1} v_{a_i}(x; \mathbf{t}; q)(s_i - \text{id}) \in \text{End}_{\mathbb{C}}(\mathcal{A})$$

for  $i = 0, \dots, n$  uniquely extends to a representation  $\pi_{\mathbf{t}, q} : H(R; \mathbf{k}) \rightarrow \text{End}_{\mathbb{C}}(\mathcal{A})$ .

The commutation relations (3.4) play a crucial role in the proof of theorem 3.2, compare with the argument in [2] and [18, (4.6)] in case of reduced root systems.

We write  $T_i$  for the image of  $T_i \in H(R; \mathbf{k})$  under the Noumi representation  $\pi_{\mathbf{t}, q}$  for  $i = 0, \dots, n$  if no confusion is possible, and we call them the difference-reflection operators associated with  $S$ .

We are now in a position to recall Sahi's [24] definition of the double affine Hecke algebra.

**Definition 3.3.** *The double affine Hecke algebra  $\mathcal{H} = \mathcal{H}(S; \mathbf{t}; q)$  is the sub-algebra of  $\text{End}_{\mathbb{C}}(\mathcal{A})$  generated by  $\pi_{\mathbf{t}, q}(H(R; \mathbf{k}))$  and  $\mathcal{A}$ , where the elements in  $\mathcal{A}$  are considered as multiplication operators in  $\text{End}_{\mathbb{C}}(\mathcal{A})$ .*

We end this section by giving an alternative presentation of  $\mathcal{H}$  (also different from Sahi's [24, sect. 3] presentation), which emphasizes its close connection with the affine root system  $S$ .

We write  $f(z) = \sum_{\lambda} c_{\lambda} z^{\lambda} \in \text{End}_{\mathbb{C}}(\mathcal{A})$  for the multiplication operator associated with the Laurent polynomial  $f(x) = \sum_{\lambda} c_{\lambda} x^{\lambda} \in \mathcal{A}$ . In particular,  $z^{\lambda+m\delta} = q^m z^{\lambda}$  ( $\lambda \in \Lambda$ ,  $m \in \frac{1}{2}\mathbb{Z}$ ) is the multiplication operator associated with  $x^{\lambda+m\delta} = q^m x^{\lambda} \in \mathcal{A}$ . Then in  $\mathcal{H}$  we have the commutation relations

$$f(z) T_i - T_i (s_i f)(z) = \frac{(t_{a_i} - t_{a_i}^{-1}) + (t_{a_i/2} - t_{a_i/2}^{-1}) z^{a_i/2}}{1 - z^{a_i}} (f(z) - (s_i f)(z)) \quad (3.8)$$

for  $i = 0, \dots, n$  and  $f \in \mathcal{A}$ . This follows from the fact that the difference-reflection operator  $T_i$  can be rewritten as

$$T_i = t_{a_i} s_i + \frac{(t_{a_i} - t_{a_i}^{-1}) + (t_{a_i/2} - t_{a_i/2}^{-1})x^{a_i/2}}{1 - x^{a_i}}(\text{id} - s_i)$$

for  $i = 0, \dots, n$ .

**Theorem 3.4.** *The double affine Hecke algebra  $\mathcal{H}(S; \mathbf{t}; q)$  is isomorphic to the unital, associative algebra  $\mathcal{F}(\mathbf{t}; q)$  with generators  $V_0^\vee, V_0, V_i$  ( $i = 1, \dots, n$ ) and  $V_n^\vee$ , satisfying*

1. *The  $H(R; \mathbf{k})$ -relations for  $(V_0, V_1, \dots, V_{n-1}, V_n)$ .*
2. *The  $H(R^\vee; \mathbf{k}^\vee)$ -relations for  $(V_0^\vee, V_1, \dots, V_{n-1}, V_n^\vee)$ .*
3. *(Compatibility conditions).  $V_n^\vee V_n V_{n-1} \cdots V_1 V_0 V_0^\vee V_1 V_2 \cdots V_{n-1} = q^{-1/2}$  and  $[V_0, V_n^\vee] = 0 = [V_0^\vee, V_n]$ .*

*The algebra isomorphism  $\phi : \mathcal{F}(\mathbf{t}; q) \rightarrow \mathcal{H}(S; \mathbf{t}; q)$  is explicitly given by  $\phi(V_i) = T_i$  ( $i = 0, \dots, n$ ),  $\phi(V_0^\vee) = T_0^{-1} z^{-a_0^\vee} = q^{-1/2} T_0^{-1} z_1$  and  $\phi(V_n^\vee) = z^{-a_n^\vee} T_n^{-1} = z_n^{-1} T_n^{-1}$ .*

*Proof.* For the existence of  $\phi$ , we need to check that the elements  $T_0^\vee = T_0^{-1} z^{-a_0^\vee}$ ,  $T_j$  ( $j = 0, \dots, n$ ) and  $T_n^\vee = z^{-a_n^\vee} T_n^{-1}$  in  $\mathcal{H}$  respect the defining relations of the generators  $V_0^\vee, V_j$  ( $j = 0, \dots, n$ ),  $V_n^\vee$  in  $\mathcal{F}$ . The  $H(R; \mathbf{k})$ -relations for  $(T_0, \dots, T_n)$  is precisely the content of theorem 3.2. The  $H(R^\vee; \mathbf{k}^\vee)$ -relations for the  $(n+1)$ -tuple  $(T_0^\vee, T_1, \dots, T_{n-1}, T_n^\vee)$  follows easily from (3.8) (see also [24, sect. 3]). By (3.8) it follows inductively that

$$\begin{aligned} z_i &= T_i^{-1} T_{i+1}^{-1} \cdots T_n^{-1} (T_n^\vee)^{-1} T_{n-1}^{-1} \cdots T_i^{-1} \\ &= q^{1/2} T_{i-1} \cdots T_1 T_0 T_0^\vee T_1 \cdots T_{i-1} \end{aligned}$$

for  $i = 1, \dots, n$  in  $\mathcal{H}$ . In particular, the identity  $z_n^{-1} z_n = 1$  in  $\mathcal{H}$  shows that the generators  $T_0^\vee, T_j$  ( $j = 0, \dots, n$ ) and  $T_n^\vee$  satisfy the compatibility condition in  $\mathcal{H}$ . Hence the algebra homomorphism  $\phi$  exists.

We write  $w_i = q^{1/2} V_{i-1} \cdots V_1 V_0 V_0^\vee V_1 \cdots V_{i-1} \in \mathcal{F}$  for  $i \in \{1, \dots, n\}$ , so that  $\phi(w_i) = z_i$  for  $i = 1, \dots, n$ . Since the  $z_i$  ( $i = 1, \dots, n$ ) and the  $T_j = \phi(V_j)$  ( $j = 0, \dots, n$ ) generate  $\mathcal{H}$  as an algebra, we see that  $\phi$  is surjective. On the other hand, all fundamental relations of  $\mathcal{H}(S; \mathbf{t}; q)$  as given by Sahi [26, sect. 3] can be easily checked for the generators  $w_i$  ( $i = 1, \dots, n$ ) and  $V_j$  ( $j = 0, \dots, n$ ) of  $\mathcal{F}$ . This implies the injectivity of  $\phi$ .  $\square$

**Remark 3.5.** The presentation of  $\mathcal{H}(S; \mathbf{t}; q)$  as given in theorem 3.4 clearly reflects the structure of the underlying non-reduced affine root system  $S$ . In particular, the first part of the compatibility condition can be recovered from the root data as follows. We put the simple roots for the indivisible roots  $R^\vee$  (respectively for the inmultipliable roots  $R$ ) above (respectively below) the corresponding vertices of the extended Dynkin diagram:

$$\begin{array}{ccccccc} a_0^\vee & & a_1 & & a_2 & & \cdots & & a_{n-1} & & a_n^\vee \\ \circ & & \circ & & \circ & & & & \circ & & \circ \\ a_0 & & a_1 & & a_2 & & & & a_{n-1} & & a_n \end{array}$$

Now we attach  $V_i (= T_i)$  to the simple roots  $a_i$ ,  $V_0^\vee (= T_0^\vee = T_0^{-1} z^{-a_0^\vee})$  to the co-root  $a_0^\vee$  and  $V_n^\vee (= T_n^\vee = z^{-a_n^\vee} T_n^{-1})$  to the co-root  $a_n^\vee$  in the above diagram. We call them the *simple generators* of  $\mathcal{F} \simeq \mathcal{H}$ . Then the compatibility condition amounts to the following rule: *multiplying simple generators in the order of appearance of*

a single walk around the diagram in clockwise direction, gives  $q^{-1/2}$ . The point of departure for the walk is irrelevant, since the compatibility condition is equivalent to the compatibility condition in which the factors in its left-hand side are permuted cyclically.

#### 4. NON-SYMMETRIC KOORNWINDER POLYNOMIALS AND TRIANGULARITY

In this section we show that the  $Y$ -operators  $Y^\lambda$  ( $\lambda \in \Lambda$ ) act as triangular operators under the Noumi representation  $\pi_{\mathbf{t},q}$ . We use this triangularity property to redefine Sahi's [24] non-symmetric Koornwinder polynomials. The advantage of this method is that triangularity properties of the non-symmetric Koornwinder polynomials are automatically incorporated in their definition, in contrast with Sahi's [24], [25] approach.

Let  $\lambda^+ \in \Lambda^+$  for  $\lambda \in \Lambda$  be the unique dominant weight in the orbit  $W\lambda$ . We will be needing the following two partial orders on the weight lattice  $\Lambda$ .

**Definition 4.1.** Let  $\lambda, \mu \in \Lambda$ .

- (i) We write  $\lambda \leq \mu$  if  $\mu - \lambda \in Q^{\vee,+}$  (and  $\lambda < \mu$  if  $\lambda \leq \mu$  and  $\lambda \neq \mu$ ).
- (ii) We write  $\lambda \preceq \mu$  if  $\lambda^+ < \mu^+$ , or if  $\lambda^+ = \mu^+$  and  $\lambda \leq \mu$  (and  $\lambda \prec \mu$  if  $\lambda \preceq \mu$  and  $\lambda \neq \mu$ ).

**Lemma 4.2.** Let  $\mu \in \Lambda$  and  $\alpha \in \Sigma^+$ .

If  $\langle \mu, \alpha \rangle \geq 2$ , then  $\mu - r\alpha^\vee \prec \mu$  for  $r = 1, \dots, \langle \mu, \alpha \rangle - 1$ .

If  $\langle \mu, \alpha \rangle \leq -2$ , then  $\mu + r\alpha^\vee \prec \mu$  for  $r = 1, \dots, -\langle \mu, \alpha \rangle - 1$ .

*Proof.* We write  $m_\alpha = \langle \mu, \alpha \rangle \in \mathbb{Z}$ . Suppose that  $m_\alpha \geq 2$  and write  $\mu_r = \mu - r\alpha^\vee$  with  $r \in \{1, \dots, m_\alpha - 1\}$ . We show that  $\mu_r^+ < \mu^+$ .

Let  $w \in W$  such that  $\mu_r^+ = w\mu_r$ . If  $w\alpha^\vee \in Q^{\vee,+}$ , then  $\mu_r^+ = w\mu - rw\alpha^\vee < w\mu \leq \mu^+$ . On the other hand, if  $w\alpha^\vee \in -Q^{\vee,+}$ , then  $\mu_r^+ = w\mu - rw\alpha^\vee < w\mu - m_\alpha w\alpha^\vee = (ws_\alpha)\mu \leq \mu^+$ . This proves the assertion for  $m_\alpha \geq 2$ . The case  $m_\alpha \leq -2$  can be obtained by applying the previous case to  $s_\alpha\mu$ .  $\square$

For  $\beta \in R$  we define

$$\mathcal{R}(\beta) = t_\beta s_\beta + t_\beta^{-1} v_\beta(x) (1 - s_\beta) \in \text{End}_{\mathbb{C}}(\mathcal{A}), \quad (4.1)$$

where  $v_\beta(\cdot)$  is given by (3.7). Let  $\epsilon : \mathbb{Z} \rightarrow \{\pm 1\}$  be the function which maps a positive integer to 1 and a strictly negative integer to -1.

**Lemma 4.3.** Let  $\lambda \in \Lambda$ . For  $\beta = \alpha + m\delta \in R^+$  with  $\alpha \in \Sigma^+$  we have

$$\mathcal{R}(\beta)(x^\lambda) = t_\beta^{\epsilon(\langle \lambda, \beta \rangle)} x^\lambda + \sum_{\mu \prec \lambda} c_{\lambda, \mu} x^\mu$$

for certain constants  $c_{\lambda, \mu} \in \mathbb{C}$ .

*Proof.* Let  $D_\beta \in \text{End}_{\mathbb{C}}(\mathcal{A})$  for  $\beta \in R$  be the divided difference-reflection operator defined by

$$D_\beta f = \frac{f - s_\beta f}{1 - x^\beta}, \quad f \in \mathcal{A}.$$

Then for all  $\lambda \in \Lambda$  we have

$$D_\beta(x^\lambda) = \begin{cases} -x^{\lambda-\beta} - x^{\lambda-2\beta} - \dots - x^{\lambda-\langle \lambda, \beta^\vee \rangle \beta} & \text{if } \langle \lambda, \beta^\vee \rangle > 0, \\ 0 & \text{if } \langle \lambda, \beta^\vee \rangle = 0, \\ x^\lambda + x^{\lambda+\beta} + \dots + x^{\lambda-(1+\langle \lambda, \beta^\vee \rangle)\beta} & \text{if } \langle \lambda, \beta^\vee \rangle < 0. \end{cases}$$



The proof follows now easily from lemma 4.2 and from the definition (4.1) of  $\mathcal{R}(\beta)$ .  $\square$

Observe that  $\mathcal{R}(a_i) = T_i s_i$  for  $i = 0, \dots, n$  and that  $\mathcal{R}(w(\beta)) = w\mathcal{R}(\beta)w^{-1}$  for all  $w \in \mathcal{W}$  and all  $\beta \in R$ . Combined with (3.5) and (3.6), we obtain

$$\begin{aligned} Y_i = & \mathcal{R}(\epsilon_i - \epsilon_{i+1})\mathcal{R}(\epsilon_i - \epsilon_{i+2}) \cdots \mathcal{R}(\epsilon_i - \epsilon_n)\mathcal{R}(2\epsilon_i) \\ & \cdot \mathcal{R}(\epsilon_i + \epsilon_n) \cdots \mathcal{R}(\epsilon_i + \epsilon_{i+1})\mathcal{R}(\epsilon_i + \epsilon_{i-1}) \cdots \mathcal{R}(\epsilon_i + \epsilon_1) \\ & \cdot \mathcal{R}(\delta + 2\epsilon_i)\tau(\epsilon_i)\mathcal{R}(\epsilon_1 - \epsilon_i)^{-1} \cdots \mathcal{R}(\epsilon_{i-1} - \epsilon_i)^{-1} \end{aligned} \quad (4.2)$$

for  $i = 1, \dots, n$ , cf. [21]. Hence the triangularity of the factors  $\mathcal{R}(\cdot)$  in (4.2) (see lemma 4.3) implies the triangularity of  $Y_i$  for  $i = 1, \dots, n$ , and hence of  $Y^\lambda$  for all  $\lambda \in \Lambda$ . For the explicit description of the diagonal terms of the  $Y$ -operators, we need to introduce some additional notations first.

We write  $f(y)$  for the value of  $f \in \mathcal{A}$  at  $y = (y_1, \dots, y_n) \in (\mathbb{C} \setminus \{0\})^n$ . In particular,  $(y)^{\lambda+m\delta} = q^m(y)^\lambda$  ( $m \in \frac{1}{2}\mathbb{Z}$ ,  $\lambda \in \Lambda$ ) is the value of  $x^{\lambda+m\delta} = q^m x^\lambda \in \mathcal{A}$  at  $y$ . Conversely, we let  $c^\lambda \in (\mathbb{C} \setminus \{0\})^n$  for  $c \in \mathbb{C} \setminus \{0\}$  and  $\lambda \in \Lambda$  be the vector  $c^\lambda = (c^{\lambda_1}, \dots, c^{\lambda_n})$ , where  $\lambda_i = \langle \lambda, \epsilon_i \rangle$ .

*Remark 4.4.* The brackets in the notation  $(y)^{\lambda+m\delta}$  for the value of  $x^{\lambda+m\delta}$  at  $y \in (\mathbb{C} \setminus \{0\})^n$  will occasionally be omitted when  $m = 0$ . For  $m \neq 0$  the brackets are needed to distinguish the value of  $x^{\lambda+m\delta}$  at  $y^{-1} = (y_1^{-1}, \dots, y_n^{-1})$  from the value of  $x^{-\lambda-m\delta}$  at  $y$ .

Let now  $\Sigma_m^+$  (respectively  $\Sigma_l^+$ ) be the positive roots in  $\Sigma$  of squared length 2 (respectively 4). Let

$$\rho_m = 2 \sum_{i=1}^n (n-i)\epsilon_i, \quad \rho_l = \sum_{i=1}^n \epsilon_i$$

be the sum of co-roots  $\alpha^\vee$  with  $\alpha \in \Sigma_m^+$  and  $\alpha \in \Sigma_l^+$ , respectively. Let  $\lambda \in \Lambda$  and set

$$\rho_m(\lambda) = \sum_{\alpha \in \Sigma_m^+} \epsilon(\langle \lambda, \alpha \rangle) \alpha^\vee, \quad \rho_l(\lambda) = \sum_{\alpha \in \Sigma_l^+} \epsilon(\langle \lambda, \alpha \rangle) \alpha^\vee.$$

Then  $\rho_m(\lambda) = \rho_m$  and  $\rho_l(\lambda) = \rho_l$  for all dominant weights  $\lambda \in \Lambda^+$ . We define now  $\gamma_\lambda = \gamma_\lambda(\mathbf{k}, q) \in \mathbb{C}^n$  by

$$\gamma_\lambda = t_0^{\rho_l(\lambda)} t_n^{\rho_l(\lambda)} t^{\rho_m(\lambda)} q^\lambda, \quad \lambda \in \Lambda, \quad (4.3)$$

where the product is the usual dot-product in  $\mathbb{C}^n$ . In particular,

$$\gamma_\lambda = (t_0 t_n t^{2(n-1)} q^{\lambda_1}, t_0 t_n t^{2(n-2)} q^{\lambda_2}, \dots, t_0 t_n q^{\lambda_n}), \quad \lambda \in \Lambda^+. \quad (4.4)$$

Then lemma 4.3, (4.2) and (2.3) lead to the following proposition.

**Proposition 4.5.** *For  $\lambda \in \Lambda$  and  $f(Y) \in \mathcal{A}_Y$ , we have*

$$f(Y)(x^\lambda) = f(\gamma_\lambda) x^\lambda + \sum_{\mu \prec \lambda} c_{\lambda, \mu} x^\mu$$

for certain constants  $c_{\lambda, \mu}$ .

We give now some properties of the diagonal terms  $\gamma_\lambda$  ( $\lambda \in \Lambda$ ) which will be used frequently in the remainder of the paper. First of all, the diagonal terms of

the  $Y$ -operators can be related to the spectrum of the  $Y$ -operators as described in [24, def. 2.5] by observing that

$$\rho_m(\lambda) = w_\lambda \rho_m, \quad \rho_l(\lambda) = w_\lambda \rho_l, \quad \lambda \in \Lambda, \quad (4.5)$$

where  $w_\lambda \in W$  is the unique element of minimal length such that  $\lambda^+ = w_\lambda^{-1} \lambda$ , see [23, prop. 2.10].

Secondly, the action of  $\mathcal{W}$  on the diagonal terms  $\gamma_\lambda$  ( $\lambda \in \Lambda$ ), induced from the dot-action of  $\mathcal{W}$  on  $\Lambda$ , is compatible with the action of  $\mathcal{W}$  on  $\mathcal{A}$  in the following way. We refer to [24, thm. 5.3] for the proof.

**Lemma 4.6.** *Let  $f \in \mathcal{A}$ . Let  $\lambda \in \Lambda$  and  $i \in \{0, 1, \dots, n\}$  such that  $s_i \cdot \lambda \neq \lambda$ . Then  $f(\gamma_{s_i \cdot \lambda}^{-1}) = (s_i f)(\gamma_\lambda^{-1})$ . If furthermore  $i \geq 1$ , then also  $f(\gamma_{s_i \cdot \lambda}) = f(\gamma_{s_i \cdot \lambda}) = (s_i f)(\gamma_\lambda)$ .*

*Remark 4.7.* Observe that the condition  $s_i \cdot \lambda \neq \lambda$  in lemma 4.6 is always met for  $i = 0$ . Let now  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda$  with  $s_i \cdot \lambda (= s_i \lambda) = \lambda$ . Then we have  $\gamma_\lambda^{a_i} = \gamma_0^{a_i}$  ( $= t^2$  if  $i < n$  and  $= t_0^2 t_n^2$  if  $i = n$ ). Hence  $(s_i f)(\gamma_\lambda^{\pm 1}) = f(\gamma_{\lambda, i}^{\pm 1})$  for  $f \in \mathcal{A}$ , with  $\gamma_{\lambda, i} = \gamma_\lambda \cdot (\gamma_0^{-a_i})^{a_i^\vee}$ . Observe that  $\gamma_{\lambda, i} \neq \gamma_\mu$  for all  $\mu \in \Lambda$  by the generic conditions on the parameters.

By lemma 4.6 we have  $f(\gamma_\lambda) = (w_\lambda^{-1} f)(\gamma_{\lambda^+})$  for  $f \in \mathcal{A}$  and  $\lambda \in \Lambda$ . Combined with (4.4) we conclude that the diagonal terms  $\gamma_\lambda$  ( $\lambda \in \Lambda$ ) are mutually different for generic values of  $q$  and  $\mathbf{k}$ . This leads to the following main result of this section.

**Theorem 4.8.** *There exists a unique basis  $\{P_\lambda\}_{\lambda \in \Lambda}$  of  $\mathcal{A}$  such that*

- $P_\lambda(x) = x^\lambda + \sum_{\mu \prec \lambda} c_{\lambda, \mu} x^\mu$  for certain constants  $c_{\lambda, \mu}$ ,
- $f(Y)P_\lambda = f(\gamma_\lambda)P_\lambda$  for all  $f(Y) \in \mathcal{A}_Y$ ,

for all  $\lambda \in \Lambda$ .

**Definition 4.9.** *The Laurent polynomial  $P_\lambda(\cdot) = P_\lambda(\cdot; \mathbf{t}; q)$  ( $\lambda \in \Lambda$ ) is called the monic, non-symmetric Koornwinder polynomial of degree  $\lambda$ .*

The terminology introduced in definition 4.9 stems from the close connection between the Laurent polynomials  $P_\lambda$  ( $\lambda \in \Lambda$ ) and Koornwinder's [14] multivariable analogues of the Askey-Wilson polynomials, see [21] and [24], as well as section 5 and section 6.

*Remark 4.10.* (i) The second property of theorem 4.8 already characterizes the non-symmetric Koornwinder polynomial  $P_\lambda$  up to a constant. This characterizing property was used by Sahi [24, def. 6.1] to introduce the non-symmetric Koornwinder polynomials. The triangularity of the non-symmetric Koornwinder polynomials was derived by Sahi [25, sect. 6] using recursion formulas.

(ii) If one uses Macdonald's [18] and Cherednik's [2]–[6] convention for the translation operator  $\tau$  (see remark 2.1), then the role of  $Y^\lambda = T_{\tau(\lambda)}$  is taken over by  $T_{\tau(\sigma\lambda)} = T_\sigma^{-1} Y^\lambda T_\sigma$  for  $\lambda \in \Lambda^+$ , where  $\sigma \in W$  is the longest Weyl group element. The common eigenfunctions then become  $T_\sigma^{-1} P_\lambda$  ( $\lambda \in \Lambda$ ). From lemma 4.3 and theorem 4.8 it follows that the  $T_\sigma^{-1} P_\lambda$  are triangular with respect to the partial order on  $\Lambda$  in which  $\mu$  is less than  $\nu$  iff  $\sigma\mu \prec \sigma\nu$  (i.e. the anti-dominant weight is highest in each  $W$ -orbit). This is in accordance with the triangular structure of non-symmetric Macdonald polynomials, see [18] and [4].

## 5. SYMMETRIC KOORNWINDER POLYNOMIALS

In this section we recall Noumi's [21] results on the affine Hecke algebraic characterization of Koornwinder's [14] multivariable analogues of the Askey-Wilson polynomials. We present Noumi's results here in a different order by making use of the triangularity of the  $Y$ -operators, see proposition 4.5.

We write  $\mathcal{A}^W = \{f \in \mathcal{A} \mid wf = f \ \forall w \in W\}$ , and similarly  $\mathcal{A}_Y^W$ , where the action is given by  $w(Y^\lambda) = Y^{w\lambda}$  for  $w \in W$  and  $\lambda \in \Lambda$ . A linear basis of  $\mathcal{A}^W$  and  $\mathcal{A}_Y^W$  is given by the monomials  $m_\lambda(x) = \sum_{\mu \in W\lambda} x^\mu$ , respectively  $m_\lambda(Y) = \sum_{\mu \in W\lambda} Y^\mu$  ( $\lambda \in \Lambda^+$ ).

It follows from (3.4) that  $\mathcal{A}_Y^W$  lies in the center  $\mathcal{Z}(H)$  of  $H$ . In fact, by [15, prop. 3.11] we know that  $\mathcal{Z}(H) = \mathcal{A}_Y^W$  (which also follows from results in section 6). The action of  $\mathcal{A}_Y^W$  on  $\mathcal{A}$  through the Noumi representation  $\pi_{t,q}$  preserves  $\mathcal{A}^W$ , compare e.g. with [18, 4.8].

We extend the action of  $\mathcal{W}$  on  $\mathcal{A}$  to an action on the quotient field  $\mathbb{C}(x) = \text{Quot}(\mathcal{A})$  by requiring  $w \in \mathcal{W}$  to be an automorphism of  $\mathbb{C}(x)$ . Let  $\mathbb{C}(x)[\mathcal{W}] \subset \text{End}_{\mathbb{C}}(\mathbb{C}(x))$  be the subalgebra generated by  $\mathbb{C}(x)$  (acting as multiplication operators) and by  $\mathcal{W}$ . Observe that  $\mathcal{H} \subset \mathbb{C}(x)[\mathcal{W}]$ , and that

$$\mathbb{C}(x)[\mathcal{W}] = \bigoplus_{w \in \mathcal{W}} \mathbb{C}(x)w = \bigoplus_{w \in W, \lambda \in \Lambda} \mathbb{C}(x)\tau(\lambda)w$$

as a  $\mathbb{C}(x)$ -submodule of  $\text{End}_{\mathbb{C}}(\mathbb{C}(x))$ , see the proof of [24, thm. 3.2]. Furthermore,  $\mathbb{C}(x)[\tau(\Lambda)] = \bigoplus_{\lambda \in \Lambda} \mathbb{C}(x)\tau(\lambda)$  is the subalgebra of  $\mathbb{C}(x)[\mathcal{W}]$  consisting of  $q$ -difference operators with coefficients in  $\mathbb{C}(x)$ .

With  $D \in \mathbb{C}(x)[\mathcal{W}]$ , say

$$D = \sum_{w \in W} D(x, w)w, \quad D(x, w) \in \mathbb{C}(x)[\tau(\Lambda)],$$

we associate a  $q$ -difference operator by

$$D_{sym} = \sum_{w \in W} D(x, w) \in \mathbb{C}(x)[\tau(\Lambda)].$$

Observe that  $Df = D_{sym}f$  if  $f \in \mathbb{C}(x)$  is  $W$ -invariant. Proposition 4.5 and lemma 4.6 imply that the  $q$ -difference operators  $f(Y)_{sym}$  ( $f \in \mathcal{A}^W$ ) are triangular endomorphisms of  $\mathcal{A}^W$ :

$$f(Y)_{sym} m_\lambda = f(\gamma_\lambda) m_\lambda + \sum_{\mu \in \Lambda^+ : \mu < \lambda} c_{\lambda, \mu} m_\mu, \quad f \in \mathcal{A}^W, \lambda \in \Lambda^+$$

for certain constants  $c_{\lambda, \mu} \in \mathbb{C}$ . This immediately implies the following result.

**Theorem 5.1.** *There exists a unique basis  $\{P_\lambda^+\}_{\lambda \in \Lambda^+}$  of  $\mathcal{A}^W$  such that*

$$\begin{aligned} - P_\lambda^+ &= m_\lambda + \sum_{\mu \in \Lambda^+ : \mu < \lambda} c_{\lambda, \mu} m_\mu \text{ for certain constants } c_{\lambda, \mu}, \\ - f(Y)_{sym} P_\lambda^+ &= f(\gamma_\lambda) P_\lambda^+ \text{ for all } f(Y) \in \mathcal{A}_Y^W, \end{aligned}$$

for all  $\lambda \in \Lambda^+$ .

Noumi [21] identified the  $q$ -difference operator

$$m_{\epsilon_1}(Y)_{sym} = (Y_1 + \cdots + Y_n + Y_1^{-1} + \cdots + Y_n^{-1})_{sym} \in \mathbb{C}(x)[\tau(\Lambda)]$$

with Koornwinder's [14] second order  $q$ -difference operator. Explicitly, Noumi [21] showed that the  $q$ -difference operator  $L = m_{\epsilon_1}(Y)_{sym} - m_{\epsilon_1}(\gamma_0)$  is given by

$$L = \sum_{j=1}^n (\phi_j^+(x)(\tau(\epsilon_j) - 1) + \phi_j^-(x)(\tau(-\epsilon_j) - 1)) \quad (5.1)$$

with  $\phi_j^-(x) = \phi_j^+(x_1^{-1}, \dots, x_n^{-1})$  and

$$\phi_j^+(x) = (t_0 t_n)^{-1} t^{2(1-n)} \frac{(1 - ax_j)(1 - bx_j)(1 - cx_j)(1 - dx_j)}{(1 - x_j^2)(1 - qx_j^2)} \cdot \prod_{i \neq j} \frac{(1 - t^2 x_i x_j)(1 - t^2 x_i^{-1} x_j)}{(1 - x_i x_j)(1 - x_i^{-1} x_j)}.$$

Here  $\{a, b, c, d\}$  is related to the multiplicity function  $\mathbf{t}$  by

$$\{a, b, c, d\} = \{t_0 t_0^\vee q^{1/2}, -t_0 (t_0^\vee)^{-1} q^{1/2}, t_n t_n^\vee, -t_n (t_n^\vee)^{-1}\}. \quad (5.2)$$

Since the spectrum of  $L \in \text{End}_{\mathbb{C}}(\mathcal{A}^W)$  is already simple (see [14]), this result implies that the  $W$ -invariant Laurent polynomials  $P_\lambda^+$  ( $\lambda \in \Lambda^+$ ) coincide with Koornwinder's [14] multivariable analogues of the Askey-Wilson polynomials.

**Definition 5.2.** *The  $W$ -invariant Laurent polynomial  $P_\lambda^+(\cdot) = P_\lambda^+(\cdot; \mathbf{t}; q)$  ( $\lambda \in \Lambda^+$ ) is called the monic, symmetric Koornwinder polynomial of degree  $\lambda \in \Lambda^+$ .*

## 6. THE ACTION OF THE HECKE ALGEBRA OF TYPE $C_n$

We associate a dual multiplicity function  $\tilde{\mathbf{t}}$  with the multiplicity function  $\mathbf{t}$  by interchanging the value of  $\mathbf{t}$  on the  $\mathcal{W}$ -orbit  $\mathcal{W}a_0$  with its value on the  $\mathcal{W}$ -orbit  $\mathcal{W}a_n^\vee$ . In other words,  $\tilde{\mathbf{t}}$  is the unique multiplicity function of  $S$  satisfying

$$\tilde{t}_0 = t_n^\vee, \quad \tilde{t}_0^\vee = t_0^\vee, \quad \tilde{t} = t, \quad \tilde{t}_n^\vee = t_0, \quad \tilde{t}_n = t_n.$$

We write  $\tilde{\mathbf{k}}$  and  $\tilde{\mathbf{k}}^\vee$  for the associated multiplicity functions of  $R$  and  $R^\vee$  respectively, and  $\tilde{v}_\beta(x) = v_\beta(x; \tilde{\mathbf{t}}; q)$  for the function  $v_\beta(\cdot)$  (3.7) with respect to dual parameters. Observe that Lusztig's formulas (3.4) can now be written in a uniform way:

$$T_i f(Y) - (s_i f)(Y) T_i = ((\tilde{t}_{a_i} - \tilde{t}_{a_i}^{-1}) + (\tilde{t}_{a_i/2} - \tilde{t}_{a_i/2}^{-1}) Y^{-a_i/2}) \left( \frac{f(Y) - (s_i f)(Y)}{1 - Y^{-a_i}} \right) \quad (6.1)$$

for  $i = 1, \dots, n$  and  $f \in \mathcal{A}$ . In the following proposition we expand  $T_i P_\lambda$  as a linear combination of non-symmetric Koornwinder polynomials.

**Proposition 6.1.** *Let  $i \in \{1, \dots, n\}$  and  $\lambda \in \Lambda$ . Then*

$$T_i P_\lambda = \xi_i(\gamma_\lambda) P_\lambda + \eta_i(\gamma_\lambda) P_{s_i \lambda} \quad (6.2)$$

with

$$\xi_i(x) = \tilde{t}_i - \tilde{t}_i^{-1} \tilde{v}_{-a_i}(x) = \frac{(\tilde{t}_{a_i}^{-1} - \tilde{t}_{a_i}) x^{a_i} + (\tilde{t}_{a_i/2}^{-1} - \tilde{t}_{a_i/2}) x^{a_i/2}}{1 - x^{a_i}} \quad (6.3)$$

and

$$\eta_i(\gamma_\lambda) = \begin{cases} \tilde{t}_i, & \text{if } \langle \lambda, a_i \rangle < 0, \\ \tilde{t}_i^{-3} \tilde{v}_{a_i}(\gamma_\lambda) \tilde{v}_{-a_i}(\gamma_\lambda), & \text{if } \langle \lambda, a_i \rangle \geq 0. \end{cases} \quad (6.4)$$

*Proof.* The proof is based on the following consequence of Lusztig's formula (6.1) and theorem 4.8: let  $\lambda \in \Lambda$  and  $i \in \{1, \dots, n\}$ , then

$$(f(Y) - (s_i f)(\gamma_\lambda))T_i P_\lambda = (f(\gamma_\lambda) - (s_i f)(\gamma_\lambda))\xi_i(\gamma_\lambda)P_\lambda, \quad \forall f \in \mathcal{A}, \quad (6.5)$$

with  $\xi_i$  given by (6.3).

Suppose now first that  $\langle \lambda, a_i \rangle = 0$ , i.e. that  $s_i \lambda = \lambda$ . Using remark 4.7, we see that  $\xi_i(\gamma_\lambda) + \eta_i(\gamma_\lambda) = t_i$ , so we have to show that  $T_i P_\lambda = t_i P_\lambda$ . Now (6.5), theorem 4.8 and remark 4.7 imply that  $T_i P_\lambda$  is a constant multiple of  $P_\lambda$ . The constant multiple can be determined by computing the leading coefficient of  $T_i P_\lambda$  using the identity  $T_i = s_i \mathcal{R}(a_i)^{-1} + t_i - t_i^{-1}$  and using lemma 4.3.

Suppose now that  $\langle \lambda, a_i \rangle \neq 0$ , then (6.5), lemma 4.6 and theorem 4.8 imply that  $T_i P_\lambda$  is of the form (6.2) for some constant  $\eta_i(\gamma_\lambda)$ , with  $\xi_i$  given by (6.3). If  $\langle \lambda, a_i \rangle < 0$ , then leading term considerations using lemma 4.3 and theorem 4.8 show that  $\eta_i(\gamma_\lambda) = t_i = \tilde{t}_i$ . The expression for  $\eta_i(\gamma_\lambda)$  when  $\langle \lambda, a_i \rangle > 0$  follows now easily by applying  $T_i$  on both sides of (6.2) and using the quadratic relation  $(T_i - t_i)(T_i + t_i^{-1}) = 0$ , compare with [22, prop. 4.1] for the proof in the rank one setting.  $\square$

We write  $S_i = [T_i, Y^{a_i}] = T_i Y^{a_i} - Y^{a_i} T_i \in \mathcal{H}$  ( $i = 1, \dots, n$ ). It follows from (6.1) that the  $S_i$  satisfy the fundamental commutation relations  $f(Y)S_i = S_i(s_i f)(Y)$  for all  $f \in \mathcal{A}$ , cf. [24, sect. 5]. We call  $S_i$  the (non-affine) intertwiner associated with the simple reflection  $s_i$ .

We use here a slightly different definition for the intertwiners  $S_i$  compared with Sahi's [24], [25] intertwiners. The advantage of the present definition is that the  $S_i$  ( $i = 1, \dots, n$ ) satisfy the  $C_n$ -braid relations, see remark 7.6. In particular, we may write  $S_w = S_{i_1} \cdots S_{i_r}$  for a reduced expression  $w = s_{i_1} \cdots s_{i_r} \in W$ , and  $S_w$  satisfies the intertwining property  $S_w f(Y) = (w f)(Y) S_w$  for all  $f \in \mathcal{A}$ . See also the paper [20], in which yet another definition for the intertwiners  $S_i$  ( $i = 0, \dots, n$ ) is used (including a non-affine intertwiner  $S_0$ ). The intertwiners in [20], which satisfy the  $\tilde{C}_n$ -braid relations, are used to prove a Rodrigues type formula for non-symmetric Koornwinder polynomials.

With our present conventions, the action of the intertwiners on the non-symmetric Koornwinder polynomials is easily determined from proposition 6.1. We give here only the action of the intertwiner  $S_i$  corresponding to a simple reflection  $s_i$  of  $W$ .

**Corollary 6.2.**  $S_i P_\lambda = (\gamma_\lambda^{a_i} - \gamma_{s_i \lambda}^{a_i}) \eta_i(\gamma_\lambda) P_{s_i \lambda}$  for  $i = 1, \dots, n$  and  $\lambda \in \Lambda$ .

*Remark 6.3.* Proposition 6.1 refines the non-affine part of Sahi's recursion formula [25, thm. 18], while corollary 6.2 can be seen as a refinement of the non-affine part of [24, thm. 5.3]. Indeed, in [25, thm. 18] and [24, thm. 5.3] the formulas are given up to an unknown multiple constant. The unknown constants in the affine part of [25, thm. 18] and [24, thm. 5.3] can also be computed, but this requires the duality properties and the evaluation formulas for the non-symmetric Koornwinder polynomials, see proposition 7.8 and theorem 9.3.

By theorem 4.8, proposition 6.1 and corollary 6.2 we have a complete description of the action of  $H(R; \mathbf{k})$  on the non-symmetric Koornwinder polynomials under the Noumi representation  $\pi_{\mathbf{t}, q}$ . From this the  $H$ -module structure of  $\mathcal{A}$  can be described

in detail, see also Sahi [24]. The result is as follows. We write

$$\mathcal{A} = \bigoplus_{\lambda \in \Lambda^+} \mathcal{A}(\lambda), \quad \mathcal{A}(\lambda) = \text{span}\{P_\mu \mid \mu \in W\lambda\}. \quad (6.6)$$

Recall that the parameters  $\mathbf{t}$  and  $q$  are assumed to be generic.

**Theorem 6.4.** *The direct sum decomposition (6.6) is the multiplicity-free, irreducible decomposition of  $\mathcal{A}$  as a  $(\pi_{\mathbf{t},q}, H(R; \mathbf{k}))$ -module. Furthermore, (6.6) is the decomposition of  $\mathcal{A}$  into isotypical components under the action of the center  $\mathcal{Z}(H(R; \mathbf{k})) = \mathcal{A}_Y^W$ . The central character  $\chi_\lambda$  of  $\mathcal{A}(\lambda)$  is given by  $\chi_\lambda(f) = f(\gamma_\lambda)$  for  $f \in \mathcal{Z}(H(R; \mathbf{k}))$ .*

We end this section by defining anti-symmetric Koornwinder polynomials and by expanding (anti-)symmetric Koornwinder polynomials in terms of non-symmetric Koornwinder polynomials.

We associate a function  $\{t_w\}_{w \in \mathcal{W}}$  with the multiplicity function  $\mathbf{t} = \{t_\beta\}_{\beta \in S}$  by defining  $t_w = t_{i_1} \dots t_{i_r}$  for a reduced expression  $w = s_{i_1} \dots s_{i_r} \in \mathcal{W}$ .

*Remark 6.5.* Recall the well-known fact that  $R^+ \cap w^{-1}R^- = \{\beta_1, \dots, \beta_r\}$  with the  $r$  distinct positive roots  $\beta_j$  given by

$$\beta_j = s_{i_r} \dots s_{i_{j+1}} a_{i_j} \quad (j = 1, \dots, r-1), \quad \beta_r = a_{i_r}.$$

In particular, it follows that

$$t_w = \prod_{\beta \in R^+ \cap w^{-1}R^-} t_\beta, \quad w \in \mathcal{W}.$$

Restricted to the finite Weyl group  $W$ , the expression for  $t_w$  reduces to

$$t_w = \prod_{\alpha \in \Sigma^+ \cap w^{-1}\Sigma^-} t_\alpha, \quad w \in W.$$

Observe in particular that  $\tilde{t}_w = t_w$  when  $w \in W$ , where  $\{\tilde{t}_w\}_{w \in \mathcal{W}}$  is the function associated with the dual multiplicity function  $\tilde{\mathbf{t}}$ .

Let  $\chi_\pm : H_0 \rightarrow \mathbb{C}$  be the trivial and alternating character of the Hecke algebra  $H_0$ , i.e.  $\chi_\pm(T_i) = \pm t_i^{\pm 1}$  for  $i = 1, \dots, n$ . Then the corresponding mutually orthogonal, primitive idempotents are given by

$$C_\pm = \frac{1}{\sum_{w \in W} t_w^{\pm 2}} \sum_{w \in W} (\pm 1)^{l(w)} t_w^{\pm 1} T_w. \quad (6.7)$$

We define for  $\lambda \in \Lambda^+$ ,

$$\begin{aligned} \mathcal{A}_\pm(\lambda) &= \{f \in \mathcal{A}(\lambda) \mid C_\pm f = f\} \\ &= \{f \in \mathcal{A}(\lambda) \mid (T_i \mp t_i^{\pm 1})f = 0 \quad \forall i = 1, \dots, n\}. \end{aligned} \quad (6.8)$$

Observe in particular that  $\mathcal{A}_+(\lambda) = \mathcal{A}(\lambda) \cap \mathcal{A}^W$ . Let  $\Lambda^{++} = \kappa + \Lambda^+$  be the cone of regular dominant weights, where

$$\kappa = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha = \sum_{i=1}^n \omega_i. \quad (6.9)$$

**Theorem 6.6.** (i)  $\mathcal{A}_+(\lambda)$  is spanned by the symmetric Koornwinder polynomial  $P_\lambda^+$  for all  $\lambda \in \Lambda^+$ .

(ii)  $\mathcal{A}_-(\lambda)$  is one-dimensional if  $\lambda \in \Lambda^{++}$  and zero dimensional otherwise. For  $\lambda \in \Lambda^{++}$  there exists a unique  $P_\lambda^- \in \mathcal{A}_-(\lambda)$  of the form  $P_\lambda^- = x^\lambda + \sum_{\mu < \lambda} e_\mu x^\mu$  for certain constants  $e_\mu \in \mathbb{C}$ .

(iii) The expressions for  $P_\lambda^+$  ( $\lambda \in \Lambda^+$ ) and for  $P_\lambda^-$  ( $\lambda \in \Lambda^{++}$ ) as linear combinations of the non-symmetric Koornwinder polynomials  $P_\mu$  ( $\mu \in W\lambda$ ) are given by

$$P_\lambda^\pm = \sum_{\mu \in W\lambda} c_{\lambda,\mu}^\pm P_\mu$$

with the coefficients  $c_{\lambda,\mu}^\pm$  ( $\mu \in W\lambda$ ) given by

$$\begin{aligned} c_{\lambda,\mu}^\pm &= (\pm 1)^{l(w_\mu)} \tilde{t}_{w_\mu}^{-2} \prod_{\substack{\alpha \in \Sigma^+ \\ \langle \mu, \alpha \rangle < 0}} \tilde{v}_{\pm\alpha}(\gamma_\mu) \\ &= (\pm 1)^{l(w_\mu)} \tilde{t}_{w_\mu}^{-2} \prod_{\alpha \in \Sigma^+ \cap w_\mu^{-1}\Sigma^-} \tilde{v}_{\pm\alpha}(\gamma_\lambda^{-1}), \end{aligned} \quad (6.10)$$

where  $w_\mu$  is the element of minimal length in  $W$  such that  $w_\mu\lambda(=w_\mu\mu^+) = \mu$ .

*Proof.* We first introduce some notations and deduce some preliminary results which we will need for the proof.

Let  $\lambda \in \Lambda^+$  and let  $Q_\lambda^\pm = \sum_{\mu \in W\lambda} d_{\lambda,\mu}^\pm P_\mu$  be an element in  $\mathcal{A}_\pm(\lambda)$ . By proposition 6.1 we have for  $\mu \in \Lambda$  and  $i \in \{1, \dots, n\}$  that

$$(T_i \mp t_i^{\pm 1})P_\mu = \xi_i^\pm(\gamma_\mu)P_\mu + \eta_i(\gamma_\mu)P_{s_i\mu}$$

with

$$\xi_i^\pm(x) = \xi_i(x) \mp t_i^{\pm 1} = \mp \tilde{t}_i^{-1} \tilde{v}_{\mp a_i}(x). \quad (6.11)$$

It follows now from the relations  $(T_i \mp t_i^{\pm 1})Q_\lambda^\pm = 0$  that the coefficients  $d_{\lambda,\mu}^\pm$  satisfy the recurrence relations

$$d_{\lambda,s_i\mu}^\pm = \left( -\frac{\xi_i^\pm(\gamma_\mu)}{\eta_i(\gamma_{s_i\mu})} \right) d_{\lambda,\mu}^\pm \quad (6.12)$$

for  $\mu \in W\lambda$  and  $i \in \{1, \dots, n\}$  such that  $\langle \mu, a_i \rangle \neq 0$ . Let now  $\mu \in W\lambda$  and choose a reduced expression  $w_\mu = s_{i_1} \cdots s_{i_r}$ . We set

$$\mu_j = s_{i_{j+1}} \cdots s_{i_r} \lambda \quad (j = 0, \dots, r-1), \quad \mu_r = \lambda.$$

Then  $\langle \mu_j, a_{i_j} \rangle > 0$  for  $j = 1, \dots, r$  by remark 6.5. Iterating (6.12), we thus obtain

$$d_{\lambda,\mu}^\pm = d_{\lambda,\lambda}^\pm \prod_{j=1}^r \left( -\frac{\xi_{i_j}^\pm(\gamma_{\mu_j})}{\eta_{i_j}(\gamma_{\mu_{j-1}})} \right). \quad (6.13)$$

*Proof of (i)* The recurrence formula (6.13) implies that  $\dim(\mathcal{A}_+(\lambda)) \leq 1$  for all  $\lambda \in \Lambda^+$ . On the other hand, theorem 5.1 implies that the symmetric Koornwinder polynomial  $P_\lambda^+$  is a non-zero element in  $\mathcal{A}_+(\lambda)$  for all  $\lambda \in \Lambda^+$ .

*Proof of (ii)* Again by (6.13), we have  $\dim(\mathcal{A}_-(\lambda)) \leq 1$ . Let  $\lambda \in \Lambda^+ \setminus \Lambda^{++}$ . Let  $s_i \in W$  be a simple reflection in  $W$  which stabilizes  $\lambda$ . Then proposition 6.1 implies that  $T_i P_\lambda = t_i P_\lambda$ . Hence the coefficient of  $P_\lambda$  in the expansion of  $(T_i + t_i^{-1})Q_\lambda^-$  as a linear combination of the  $P_\nu$  ( $\nu \in W\lambda$ ), is  $(t_i + t_i^{-1})d_{\lambda,\lambda}^-$ . On the other hand,

$(T_i + t_i^{-1})Q_\lambda^- = 0$ , hence we conclude that  $d_{\lambda,\lambda}^- = 0$ . Then (6.13) implies  $d_{\lambda,\mu}^- = 0$  for all  $\mu \in W\lambda$ , hence  $Q_\lambda^- = 0$ . This proves that  $\mathcal{A}_-(\lambda) = \{0\}$  if  $\lambda \in \Lambda^+ \setminus \Lambda^{++}$ .

Let now  $\lambda \in \Lambda^{++}$ . Then  $C_-P_{\sigma\lambda} \in \mathcal{A}_-(\lambda)$ , where  $\sigma \in W$  is the longest Weyl group element. Furthermore, by proposition 6.1 we have  $C_-P_{\sigma\lambda} = \sum_{\mu \in W\lambda} d_\mu P_\mu$  with  $d_\lambda = (\sum_{w \in W} t_w^{-2})^{-1}(-1)^{l(\sigma)} \neq 0$ , so  $C_-P_{\sigma\lambda}$  is a non-zero element in  $\mathcal{A}_-(\lambda)$ . Hence  $\dim(\mathcal{A}_-(\lambda)) = 1$  if  $\lambda \in \Lambda^{++}$ . The triangularity statement follows from theorem 4.8 and from the fact that the coefficient  $d_\lambda$  in the above expansion of  $C_-P_{\sigma\lambda}$  is non-zero.

*Proof of (iii)* We have to show that

$$d_{\lambda,\mu}^\pm = d_{\lambda,\lambda}^\pm c_{\lambda,\mu}^\pm, \quad \mu \in W\lambda, \quad (6.14)$$

where the  $d_{\lambda,\mu}^\pm$  are the expansion coefficients of  $Q_\lambda^\pm$  in terms of non-symmetric Koornwinder polynomials  $P_\mu$  ( $\mu \in W\lambda$ ), and where  $c_{\lambda,\mu}^\pm$  is given by (6.10). We use again the recurrence formula (6.13) for the coefficients  $d_{\lambda,\mu}^\pm$ . We set

$$\alpha_1 = -a_{i_1}, \quad \alpha_j = -s_{i_1} \cdots s_{i_{j-1}} a_{i_j} \quad (j = 2, \dots, r),$$

then the  $\alpha_j$  ( $j = 1, \dots, r$ ) are mutually different and

$$\{\alpha_1, \dots, \alpha_r\} = \Sigma^- \cap w_\mu \Sigma^+ = \{\alpha \in \Sigma^- \mid \langle \mu, \alpha \rangle > 0\}, \quad (6.15)$$

see remark 6.5. Now observe that  $\langle \mu_{j-1}, a_{i_j} \rangle = -\langle \mu_j, a_{i_j} \rangle < 0$  for all  $j = 1, \dots, r$ , so that

$$\eta_{i_j}(\gamma_{\mu_{j-1}}) = \tilde{t}_{\alpha_j} \quad (j = 1, \dots, r). \quad (6.16)$$

Furthermore, observe that

$$(\gamma_{\mu_j})^{a_{i_j}} = (\gamma_\mu)^{\alpha_j} = (\gamma_\lambda)^{w_\mu^{-1}\alpha_j}, \quad j = 1, \dots, r \quad (6.17)$$

by lemma 4.6. Substituting (6.16) and (6.11) in (6.13) and using (6.17) and the characterization of the roots  $\{\alpha_j\}_{j=1}^r$  (see (6.15)), we obtain (6.14).  $\square$

**Definition 6.7.** *The Laurent polynomial  $P_\lambda^- \in \mathcal{A}_-(\lambda)$  ( $\lambda \in \Lambda^{++}$ ) is called the anti-symmetric Koornwinder polynomial of degree  $\lambda$ .*

*Remark 6.8.* (i) Part (i) of theorem 6.6 was also observed by Sahi [24, cor. 6.6].

(ii) Theorem 6.6 extends Macdonald's [18, sect. 6] explicit expansion formulas for the (anti-)symmetric Macdonald polynomials associated with root systems of classical type.

## 7. SPECTRAL DIFFERENCE-REFLECTION OPERATORS AND DUALITY

Let  $x_\lambda = \gamma_\lambda(\tilde{\mathbf{k}}, q)$  ( $\lambda \in \Lambda$ ) be the spectrum (4.3) of the  $\tilde{Y}$ -operators, and denote  $\tilde{\mathcal{H}} = \mathcal{H}(S; \tilde{\mathbf{t}}; q)$  for the double affine Hecke algebra with respect to dual parameters. We define evaluation mappings  $\text{Ev} : \mathcal{H} \rightarrow \mathbb{C}$  and  $\widetilde{\text{Ev}} : \tilde{\mathcal{H}} \rightarrow \mathbb{C}$  by

$$\text{Ev}(X) = (X(1))(x_0^{-1}), \quad \widetilde{\text{Ev}}(\tilde{X}) = (\tilde{X}(1))(\gamma_0^{-1})$$

for  $X \in \mathcal{H}$  and  $\tilde{X} \in \tilde{\mathcal{H}}$ , where  $1 \in \mathcal{A}$  is the Laurent polynomial identically equal to one.

The evaluation  $\text{Ev}(P_\lambda(z)) = P_\lambda(x_0^{-1})$  of the non-symmetric monic Koornwinder polynomial  $P_\lambda(\cdot) = P_\lambda(\cdot; \mathbf{t}; q)$  is generically non-zero by the analytic dependence of  $P_\lambda$  on  $\mathbf{t}$  and  $q$  (we use here that  $P_\lambda(x) = x^\lambda$  when  $t_a = 1$  for all  $a \in S$ ). Similarly,  $\text{Ev}(P_\lambda^+(z)) = P_\lambda^+(x_0^{\pm 1})$  is non-zero for generic parameter values  $\mathbf{t}$  and  $q$ .



In section 9 we explicitly evaluate  $P_\lambda(x_0^{-1})$  and  $P_\lambda^+(x_0) = P_\lambda^+(x_0^{-1})$ , so that the generic conditions on the parameters can be made completely explicit.

**Definition 7.1.** (i) Let  $E(\gamma_\lambda; \cdot) = E(\gamma_\lambda; \cdot; \mathbf{t}; q)$  be the constant multiple of the non-symmetric Koornwinder polynomial  $P_\lambda(\cdot)$  of degree  $\lambda \in \Lambda$  which takes the value one at  $x = x_0^{-1}$ .

(ii) Let  $E^+(\gamma_\lambda; \cdot) = E^+(\gamma_\lambda; \cdot; \mathbf{t}; q)$  be the constant multiple of the symmetric Koornwinder polynomial  $P_\lambda^+(\cdot)$  of degree  $\lambda \in \Lambda^+$  which takes the value one at  $x = x_0$ .

Sahi [24] showed that the role of the geometric parameter  $x = x_\mu$  and of the spectral parameter  $\gamma = \gamma_\lambda$  are (in a suitable sense) interchangeable for the renormalized Koornwinder polynomials  $E(\gamma; x^{-1})$  and  $E^+(\gamma; x)$ , see also van Diejen [9] for a sub-class of the symmetric Koornwinder polynomials. These duality properties stem from a particular anti-algebra isomorphism of the double affine Hecke algebra  $\mathcal{H}$ , which we define now first.

Recall the notations  $T_0^\vee = T_0^{-1} z^{-a_0^\vee} \in \mathcal{H}$  and  $T_n^\vee = z^{-a_n^\vee} T_n^{-1} \in \mathcal{H}$  for the simple generators associated with  $a_0^\vee$  and  $a_n^\vee$  respectively, see remark 3.5. We set

$$U_n = T_1 T_2 \cdots T_{n-1} T_n^\vee T_{n-1}^{-1} \cdots T_2^{-1} T_1^{-1},$$

which is a conjugate of  $T_n^\vee$  in  $\mathcal{H}(S; \mathbf{t}; q)$ .

Set  $\mathbf{t}^{-1} = (t_\beta^{-1})_{\beta \in S}$  for the inverse of the multiplicity function  $\mathbf{t}$ . We write  $T'_i, T_j^{\vee'}, U'_n, Y'^\lambda, z'^\lambda$  for the elements  $T_i, T_j^\vee, U_n, Y^\lambda$  and  $z^\lambda$  in the double affine Hecke algebra  $\mathcal{H}' = \mathcal{H}(S; \mathbf{t}^{-1}; q^{-1})$ . Similarly, we write  $\tilde{T}_i, \dots$  (respectively  $\tilde{T}'_i, \dots$ ) for the elements  $T_i, \dots$  in the double affine Hecke algebra  $\tilde{\mathcal{H}}$  (respectively  $\tilde{\mathcal{H}}' = \mathcal{H}(S; \tilde{\mathbf{t}}^{-1}; q^{-1})$ ). The following theorem was proved by Sahi [24, thm. 4.2].

**Theorem 7.2.** *There exists a unique algebra isomorphism  $\epsilon = \epsilon_{\mathbf{t}, q} : \mathcal{H} \rightarrow \tilde{\mathcal{H}}'$  satisfying  $\epsilon(T_0) = (\tilde{U}'_n)^{-1}$ ,  $\epsilon(z_i) = \tilde{Y}'_i$  and  $\epsilon(T_i) = (\tilde{T}'_i)^{-1}$  for  $i = 1, \dots, n$ . Furthermore,  $\epsilon_{\mathbf{t}, q}^{-1} = \epsilon_{\tilde{\mathbf{t}}^{-1}, q^{-1}}$ .*

The isomorphism  $\epsilon$  is a crucial building block for Sahi's [24] duality anti-isomorphism of the double affine Hecke algebra  $\mathcal{H}$ . In fact, the duality anti-isomorphism is obtained by composing  $\epsilon$  with the anti-isomorphism  $\ddagger$  defined in the following lemma.

**Lemma 7.3.** *There exists a unique algebra isomorphism  $\dagger = \dagger_{\mathbf{t}, q} : \mathcal{H} \rightarrow \mathcal{H}'$  (respectively anti-algebra isomorphism  $\ddagger = \ddagger_{\mathbf{t}, q} : \mathcal{H} \rightarrow \mathcal{H}'$ ) satisfying  $T_i \mapsto (T'_i)^{-1}$  ( $i = 0, \dots, n$ ) and  $z_j \mapsto (z'_j)^{-1}$  ( $j = 1, \dots, n$ ).*

*Proof.* This follows directly from the presentation of  $\mathcal{H}$  as given by Sahi [24, sect. 3].  $\square$

**Remark 7.4.** (i) Lemma 7.3 for  $\ddagger$  was observed by Sahi [24, prop. 7.1].

(ii) In proposition 8.3 we interpret the anti-algebra isomorphism  $\ddagger$  as a  $*$ -structure on  $\mathcal{H} \subset \text{End}_{\mathbb{C}}(\mathcal{A})$  induced from a suitable non-degenerate bilinear form on  $\mathcal{A}$ .

We write  $\tilde{\dagger}'$  (respectively  $\tilde{\ddagger}'$ ) for  $\dagger$  (respectively  $\ddagger$ ) with respect to the parameters  $(\tilde{\mathbf{t}}^{-1}, q^{-1})$ .

**Definition 7.5.** (i) The algebra isomorphism  $\Phi = \Phi_{\mathbf{t}, q} = \tilde{\dagger}' \circ \epsilon : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$  is called the duality isomorphism of  $\mathcal{H}$ .

(ii) The anti-algebra isomorphism  $\Psi = \Psi_{\mathbf{t}, q} = \tilde{\ddagger}' \circ \epsilon : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$  is called the duality anti-isomorphism of  $\mathcal{H}$ .

Observe that  $\Phi$  (respectively  $\Psi$ ) is uniquely characterized as the (anti-)algebra homomorphism  $\mathcal{H} \rightarrow \tilde{\mathcal{H}}$  which maps  $U_n$  to  $\tilde{T}_0$ ,  $T_i$  to  $\tilde{T}_i$  and  $Y_i$  to  $\tilde{z}_i^{-1}$  for  $i = 1, \dots, n$ . By [24, sect. 7], the inverse of  $\Psi = \Psi_{\mathbf{t}, q}$  is given by  $\tilde{\Psi} = \Psi_{\tilde{\mathbf{t}}, q}$ .

*Remark 7.6.* Observe that the image of the non-affine intertwiners  $S_i = [T_i, Y^{a_i}] \in \mathcal{H}$  ( $i = 1, \dots, n$ ) under  $\Psi$  is given by

$$\Psi(S_i) = \tilde{t}_i^{-1}(\tilde{z}^{-a_i} - \tilde{z}^{a_i})\tilde{v}_{a_i}(\tilde{z})s_i \in \tilde{\mathcal{H}}$$

in view of the explicit expression for  $\tilde{T}_i$  (see theorem 3.2). In particular,  $\Psi(S_i)$  is of the form  $f_i(\tilde{z}^{a_i})s_i$ , with  $f_i$  a Laurent polynomial in one variable and with  $f_1 = f_2 = \dots = f_{n-1}$ . From these facts it is easy to prove that  $(S_1, \dots, S_n)$  satisfies the  $C_n$ -braid relations in  $\mathcal{H}$ .

The two evaluation mappings  $\text{Ev}$  and  $\tilde{\text{Ev}}$  are related via the duality anti-isomorphism:

$$\tilde{\text{Ev}}(\Psi(X)) = \text{Ev}(X), \quad X \in \mathcal{H},$$

see [24, thm. 7.3]. This implies that the two pairings  $B : \mathcal{H} \times \tilde{\mathcal{H}} \rightarrow \mathbb{C}$  and  $\tilde{B} : \tilde{\mathcal{H}} \times \mathcal{H} \rightarrow \mathbb{C}$  defined by  $B(X, \tilde{X}) = \text{Ev}(\tilde{\Psi}(\tilde{X})X)$  and  $\tilde{B}(\tilde{X}, X) = \tilde{\text{Ev}}(\Psi(X)\tilde{X})$  for  $X \in \mathcal{H}$  and  $\tilde{X} \in \tilde{\mathcal{H}}$  satisfy the duality property

$$B(X, \tilde{X}) = \tilde{B}(\tilde{X}, X), \quad X \in \mathcal{H}, \tilde{X} \in \tilde{\mathcal{H}}. \quad (7.1)$$

Before we recall how (7.1) implies the duality properties of the Koornwinder polynomials, we first collect some elementary identities for the bilinear form  $B$ . The proof of the lemma is similar to the proof in the rank one setting, see [22, lem. 10.5].

**Lemma 7.7.** *Let  $f \in \mathcal{A}$ . Let  $X, X_1, X_2 \in \mathcal{H}$  and  $\tilde{X}, \tilde{X}_1, \tilde{X}_2 \in \tilde{\mathcal{H}}$ .*

(i)  $B(X_1 X_2, \tilde{X}) = B(X_2, \Psi(X_1)\tilde{X})$  and  $B(X, \tilde{X}_1 \tilde{X}_2) = B(\tilde{\Psi}(\tilde{X}_1)X, \tilde{X}_2)$ .

(ii)  $B(X T_i, \tilde{X}) = t_i B(X, \tilde{X})$  for  $i = 0, \dots, n$ .

(iii)  $B((X(f))(z), \tilde{X}) = B(Xf(z), \tilde{X})$  and  $B(X, (\tilde{X}(f))(\tilde{z})) = B(X, \tilde{X}f(\tilde{z}))$ , where  $(X(f))(z)$  is the multiplication operator in  $\mathcal{H}$  corresponding to the Laurent polynomial  $X(f) \in \mathcal{A}$ , and  $Xf(z)$  is the product of the elements  $X$  and  $f(z)$  in  $\mathcal{H}$ .

We write  $\tilde{E}(x_\lambda; \cdot)$  for the renormalized non-symmetric Koornwinder polynomial  $E(x_\lambda; \cdot; \tilde{\mathbf{t}}; q)$ , and similarly for  $\tilde{E}^+(x_\lambda; \cdot)$ . Observe now that by lemma 7.7 and theorem 4.8,

$$f(\gamma_\lambda^{-1}) = \tilde{B}(f(\tilde{z}), E(\gamma_\lambda; z)), \quad g(x_\mu^{-1}) = B(g(z), \tilde{E}(x_\mu; \tilde{z})) \quad (7.2)$$

for  $f, g \in \mathcal{A}$  and  $\lambda, \mu \in \Lambda$ . Taking  $f = \tilde{E}(x_\mu; \cdot)$  and  $g = E(\gamma_\lambda; \cdot)$  and using the duality (7.1) for the pairing, we arrive at

$$E(\gamma_\lambda; x_\mu^{-1}) = \tilde{E}(x_\mu; \gamma_\lambda^{-1}), \quad \lambda, \mu \in \Lambda \quad (7.3)$$

which is the duality for the renormalized Koornwinder polynomials, see [24, thm. 7.4]. Similarly, we derive from theorem 5.1 and (7.1) that

$$E^+(\gamma_\lambda; x_\mu) = \tilde{E}^+(x_\mu; \gamma_\lambda), \quad \lambda, \mu \in \Lambda^+, \quad (7.4)$$

see [24, cor. 7.5]. Using the duality (7.3), we can rewrite the action of  $T_i$  ( $i = 1, \dots, n$ ) and  $U_n$  on the renormalized Koornwinder polynomials  $E(\gamma; \cdot)$  in terms of difference-reflection operators acting on the spectral parameter  $\gamma \in \text{Spec}(Y) = \{\gamma_\lambda \mid \lambda \in \Lambda\}$ . Define an action of  $\mathcal{W}$  on  $\text{Spec}(Y)$  by  $w\gamma_\lambda = \gamma_{w.\lambda}$  ( $\lambda \in \Lambda$ ,  $w \in \mathcal{W}$ ).

**Proposition 7.8.** (i) For  $\gamma \in \text{Spec}(Y)$  we have

$$(U_n E(\gamma; \cdot))(x) = \tilde{t}_0 E(\gamma; x) + \tilde{t}_0^{-1} \tilde{v}_{a_0}(\gamma^{-1})(E(s_0 \gamma; x) - E(\gamma; x)).$$

(ii) For  $i = 1, \dots, n$  and  $\gamma \in \text{Spec}(Y)$  we have

$$(T_i E(\gamma; \cdot))(x) = \tilde{t}_i E(\gamma; x) + \tilde{t}_i^{-1} \tilde{v}_{a_i}(\gamma^{-1})(E(s_i \gamma; x) - E(\gamma; x)).$$

*Proof.* By (7.2) and lemma 7.7 we have

$$B(E(\gamma_\lambda; z), \tilde{T}_i \tilde{E}(x_\mu; \tilde{z})) = \begin{cases} (U_n E(\gamma_\lambda; \cdot))(x_\mu^{-1}) & \text{if } i = 0 \\ (T_i E(\gamma_\lambda; \cdot))(x_\mu^{-1}) & \text{if } i = 1, \dots, n \end{cases}$$

for all  $\lambda, \mu \in \Lambda$ . So it suffices to prove that

$$\begin{aligned} B(E(\gamma_\lambda; z), \tilde{T}_i \tilde{E}(x_\mu; \tilde{z})) &= \tilde{t}_i E(\gamma_\lambda; x_\mu^{-1}) \\ &\quad + \tilde{t}_i^{-1} \tilde{v}_{a_i}(\gamma_\lambda^{-1})(E(\gamma_{s_i \cdot \lambda}; x_\mu^{-1}) - E(\gamma_\lambda; x_\mu^{-1})) \end{aligned} \quad (7.5)$$

for all  $\lambda, \mu \in \Lambda$  and all  $i = 0, \dots, n$ .

Formula (7.5) is easy when  $\lambda$  is stabilized by  $s_i$  since then we have

$$B(E(\gamma_\lambda; z), \tilde{T}_i \tilde{E}(x_\mu; \tilde{z})) = B((T_i(E(\gamma_\lambda; \cdot)))(z), \tilde{E}(x_\mu; \tilde{z})) = \tilde{t}_i E(\gamma_\lambda; x_\mu^{-1})$$

where the last equality follows from (the proof of) proposition 6.1 and (7.2). So we assume for the remainder of the proof that  $s_i \cdot \lambda \neq \lambda$ . We can use now (3.8) to commute  $\tilde{T}_i$  and  $\tilde{E}(x_\mu; \tilde{z})$  in the left-hand side of (7.5). Combined with lemma 7.7 we then derive that

$$\begin{aligned} B(E(\gamma_\lambda; z), \tilde{T}_i \tilde{E}(x_\mu; \tilde{z})) &= \tilde{t}_i (s_i \tilde{E}(x_\mu; \cdot))(\gamma_\lambda^{-1}) \\ &\quad + \psi_i(\gamma_\lambda^{-1})(\tilde{E}(x_\mu; \gamma_\lambda^{-1}) - (s_i \tilde{E}(x_\mu; \cdot))(\gamma_\lambda^{-1})), \end{aligned}$$

where  $\psi_i \in \mathbb{C}(x)$  is given by

$$\psi_i(x) = \frac{(\tilde{t}_{a_i} - \tilde{t}_{a_i}^{-1}) + (\tilde{t}_{a_i/2} - \tilde{t}_{a_i/2}^{-1})x^{a_i/2}}{1 - x^{a_i}} = \tilde{t}_i - \tilde{t}_i^{-1} \tilde{v}_{a_i}(x).$$

Since  $s_i \cdot \lambda \neq \lambda$ , we can apply lemma 4.6 together with the duality (7.3) of the non-symmetric Koornwinder polynomials to obtain the desired formula (7.5).  $\square$

## 8. (BI-)ORTHOGONALITY RELATIONS AND QUADRATIC NORMS

From now on we assume that  $0 < q, t < 1$  and that the parameters  $a, b, c, d$  (see (5.2)) have moduli less than one. We define  $\Delta(\cdot) = \Delta(\cdot; \mathbf{t}; q)$  and  $\Delta_+(\cdot) = \Delta_+(\cdot; \mathbf{t}; q)$  by

$$\Delta(x; \mathbf{t}; q) = \prod_{\beta \in R^+} \frac{1}{v_\beta(x; \mathbf{t}; q)} \quad (8.1)$$

and

$$\begin{aligned} \Delta_+(x; \mathbf{t}; q) &= \prod_{\substack{\beta \in R \\ \beta(0) \geq 0}} \frac{1}{v_\beta(x; \mathbf{t}; q)} \\ &= \prod_{1 \leq i < j \leq n} \frac{(x_i x_j, x_i x_j^{-1}, x_i^{-1} x_j, x_i^{-1} x_j^{-1}; q)_\infty}{(t^2 x_i x_j, t^2 x_i x_j^{-1}, t^2 x_i^{-1} x_j, t^2 x_i^{-1} x_j^{-1}; q)_\infty} \\ &\quad \cdot \prod_{i=1}^n \frac{(x_i^2, x_i^{-2}; q)_\infty}{(a x_i, a x_i^{-1}, b x_i, b x_i^{-1}, c x_i, c x_i^{-1}, d x_i, d x_i^{-1}; q)_\infty}, \end{aligned} \quad (8.2)$$

where  $(y_1, \dots, y_m; q)_\infty = \prod_{j=1}^m (y_j; q)_\infty$  with  $(y; q)_\infty = \prod_{j=0}^\infty (1 - yq^j)$  the  $q$ -shifted factorial. The second equality in (8.2) follows from the  $\mathcal{W}$ -orbit structure of the reduced affine root system  $R$  (cf. (2.3)), together with (5.2). Observe that  $\Delta_+(\cdot)$  is  $W$ -invariant, where  $W$  acts by permutations and inversions of the coordinates  $x = (x_1, \dots, x_n)$ . Furthermore,

$$\Delta(x; \mathbf{t}; q) = \mathcal{C}(x; \mathbf{t}; q) \Delta_+(x; \mathbf{t}; q) \quad (8.3)$$

with  $\mathcal{C}(x) = \mathcal{C}(x; \mathbf{t}; q)$  given by

$$\mathcal{C}(x; \mathbf{t}; q) = \prod_{\alpha \in \Sigma^-} v_\alpha(x; \mathbf{t}; q). \quad (8.4)$$

We define now bilinear forms  $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathbf{t}, q}$  and  $\langle \cdot, \cdot \rangle_+ = \langle \cdot, \cdot \rangle_{+, \mathbf{t}, q}$  on  $\mathcal{A}$  by

$$\begin{aligned} \langle f, g \rangle &= \frac{1}{(2\pi i)^n} \iint_{x \in \mathbb{T}^n} f(x) (\sigma g)(x) \Delta(x) \frac{dx}{x}, \\ \langle f, g \rangle_+ &= \frac{1}{(2\pi i)^n} \iint_{x \in \mathbb{T}^n} f(x) (\sigma g)(x) \Delta_+(x) \frac{dx}{x} \end{aligned} \quad (8.5)$$

where  $\frac{dx}{x} = \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n}$  and  $\mathbb{T} \subset \mathbb{C}$  is the (positively oriented) unit circle. Recall here that  $\sigma$  is the longest Weyl group element in  $W$ . Observe that the bilinear forms  $\langle \cdot, \cdot \rangle$  and  $\langle \cdot, \cdot \rangle_+$  are non-degenerate in both factors.

The bilinear form  $\langle \cdot, \cdot \rangle_+$  coincides with Koornwinder's [14] pairing for the symmetric Koornwinder polynomials, see also [26] for an extension to more general parameter values. If on the other hand the parameter values are such that  $\Delta(\cdot) \in \mathcal{A}$ , then  $\langle f, g \rangle$  equals the constant term of the Laurent polynomial  $f(x) (\sigma g)(x) \Delta(x)$  and  $\langle \cdot, \cdot \rangle$  then coincides with Sahi's [25] pairing for the non-symmetric Koornwinder polynomials.

**Lemma 8.1.** *We have*

$$\sum_{w \in W} w \mathcal{C}(\cdot; \mathbf{t}; q) = K_{\mathbf{t}, q} \quad (8.6)$$

in  $\mathbb{C}(x)$  for some constant  $K = K_{\mathbf{t}, q} \in \mathbb{C}$ . In particular,

$$\langle f, g \rangle = \frac{K}{|W|} \langle f, g \rangle_+, \quad \forall f, g \in \mathcal{A}^W \quad (8.7)$$

where  $|W| = 2^n n!$  is the cardinality of the finite Weyl group  $W$ .

*Proof.* The first statement follows from [17, (2.8 n.r)] with the indeterminates in [17, (2.8 n.r)] specialized to  $u_\alpha^{1/2} = -t_\alpha t_{\alpha/2}$  for  $\alpha \in \Sigma_l^+$ ,  $u_\alpha = t_\alpha^2$  for  $\alpha \in \Sigma_m^+$  and  $u_\alpha = t_\alpha^{-2}$  for  $\alpha \in \frac{1}{2}\Sigma_l^+$ . The identity (8.7) follows then from (8.3) and the invariance of the measure  $(\mathbb{T}^n, \frac{dx}{x})$  under the action of  $W$ .  $\square$

A product form for the constant  $K$  can be obtained by specializing the left hand side of (8.6) at  $x_0^{-1}$ , see [17, (2.4 n.r)]. In fact, we have the following more general result.

**Lemma 8.2.** *Let  $\lambda \in \Lambda^+$ , and write  $W_\lambda$  (respectively  $W^\lambda$ ) for the stabilizer subgroup of  $\lambda$  in  $W$  (respectively the minimal coset representatives of  $W/W_\lambda$ ). Then  $K = \sum_{w \in W^\lambda} \mathcal{C}(x_{w\lambda}^{-1})$ . In particular,  $K = \mathcal{C}(x_0^{-1})$ .*

*Proof.* By the definition (8.6) of  $K$  we have

$$K = \sum_{w \in W^\lambda, u \in W_\lambda} (u^{-1}w^{-1}\mathcal{C})(x_\lambda^{-1}). \quad (8.8)$$

We consider a term  $(u^{-1}w^{-1}\mathcal{C})(x_\lambda^{-1})$  in this sum with  $u \neq 1$ . Then there exists a simple root  $a_i$  ( $i \in \{1, \dots, n\}$ ) which is orthogonal to  $\lambda$ , and which is mapped to a negative root  $\alpha$  by  $wu$ . Now remark 4.7 implies that the factor  $v_{u^{-1}w^{-1}\alpha}(x_\lambda^{-1}) = v_{a_i}(x_\lambda^{-1})$  of  $(u^{-1}w^{-1}\mathcal{C})(x_\lambda^{-1})$  is zero. Hence the contribution in the sum (8.8) is zero unless  $u = 1$ . The lemma follows now from lemma 4.6.  $\square$

**Proposition 8.3.** *For  $X \in \mathcal{H}$  we have*

$$\langle X(f), g \rangle = \langle f, X^\dagger(g) \rangle, \quad f, g \in \mathcal{A},$$

where  $\dagger : \mathcal{H} \rightarrow \mathcal{H}'$  is the anti-algebra isomorphism defined in lemma 7.3.

*Proof.* The proposition is obviously correct for  $X = z^\lambda$  ( $\lambda \in \Lambda$ ), so it suffices to prove it for  $X = T_i$  ( $i = 0, \dots, n$ ). Let  $f, g \in \mathcal{A}$ . It follows by direct computations that

$$(T_i f)(x)(\sigma g)(x) - f(x)(\sigma((T'_i)^{-1}g))(x) = t_i^{-1}h_i(x)v_{a_i}(x) \quad (8.9)$$

for  $i = 0, \dots, n$ , with

$$h_i(x) = (s_i f)(x)(\sigma g)(x) - f(x)(s_i(\sigma g))(x)$$

and with the action of  $s_i$  as defined in (3.1). Now observe that  $h_i$  is  $s_i$ -alternating, i.e.  $s_i h_i = -h_i$  for  $i = 0, \dots, n$ . On the other hand,

$$v_{a_i}(x)\Delta(x) = \prod_{\beta \in R^+ \setminus \{a_i\}} \frac{1}{v_\beta(x)} \quad (8.10)$$

is invariant under the action of  $s_i$  for  $i = 0, \dots, n$ , where the action of  $s_i$  is extended from  $\mathcal{A}$  to (suitably nice) functions  $f$  in the  $n$  variables  $x = (x_1, \dots, x_n)$  via the formulas (3.1). This is an immediate consequence of the well-known fact that the roots  $R^+ \setminus \{a_i\}$  are permuted by the simple reflection  $s_i$ . Hence  $\langle T_i f, g \rangle - \langle f, (T'_i)^{-1}g \rangle$  can be rewritten as an integral over  $(\mathbb{T}^n, \frac{dx}{x})$  with  $s_i$ -alternating integrand for all  $i \in \{0, \dots, n\}$ .

Now  $\langle T_i f, g \rangle - \langle f, (T'_i)^{-1}g \rangle = 0$  for  $i = 1, \dots, n$  follows from the fact that the measure  $(\mathbb{T}^n, \frac{dx}{x})$  is  $W$ -invariant. The case  $i = 0$  is more subtle. The behaviour of the measure  $(\mathbb{T}^n, \frac{dx}{x})$  under the action of  $s_0$  is given by

$$\iint_{x \in \mathbb{T}^n} (s_0 h)(x) \frac{dx}{x} = \iint_{y_1 \in q\mathbb{T}} \iint_{y \in \mathbb{T}^{n-1}} h(y_1, y) \frac{dy_1}{y_1} \frac{dy}{y},$$

which now implies that

$$\begin{aligned} \langle T_0 f, g \rangle - \langle f, (T'_0)^{-1}g \rangle &= \\ &= \frac{1}{2(2\pi i)^n} \int_{y_1 \in \mathbb{T} - q\mathbb{T}} \iint_{y \in \mathbb{T}^{n-1}} t_0^{-1} h_0(y_1, y) v_{a_0}(y_1, y) \Delta(y_1, y) \frac{dy_1}{y_1} \frac{dy}{y}. \end{aligned} \quad (8.11)$$

For fixed  $y \in \mathbb{T}^{n-1}$ , the integrand in the right-hand side of (8.11) depends analytically on  $y_1 \in \{v \in \mathbb{C} \mid q \leq |v| \leq 1\}$ . Indeed, by a direct computation using (5.2)

and the second expression of  $\Delta_+(x)$  in (8.2), we see that the  $y_1$ -dependent factor of  $v_{a_0}(y_1, y)\Delta(y_1, y)$  is given by

$$\frac{(y_1^2, q^2 y_1^{-2}; q)_\infty}{(ay_1, by_1, cy_1, dy_1, qay_1^{-1}, qby_1^{-1}, qcy_1^{-1}, qdy_1^{-1}; q)_\infty} \cdot \prod_{j=2}^n \frac{(y_1 y_j, y_1 y_j^{-1}, qy_1^{-1} y_j, qy_1^{-1} y_j^{-1}; q)_\infty}{(t^2 y_1 y_j, t^2 y_1 y_j^{-1}, qt^2 y_1^{-1} y_j, qt^2 y_1^{-1} y_j^{-1}; q)_\infty},$$

which has the desired analytic behaviour due to the conditions on the parameters  $q$  and  $\mathbf{t}$ . Thus by Cauchy's theorem we conclude that  $\langle T_0 f, g \rangle - \langle f, (T'_0)^{-1} g \rangle = 0$ . This completes the proof of the proposition.  $\square$

*Remark 8.4.* An algebraic proof of proposition 8.3 was given by Sahi [25, thm. 16] for those (discrete) values of  $\mathbf{t}$  such that  $\Delta(\cdot; \mathbf{t}; q) \in \mathcal{A}$ .

We write  $E'(\gamma_\lambda^{-1}; \cdot)$  for the renormalized Koornwinder polynomial of degree  $\lambda \in \Lambda$  with respect to inverse parameters  $(\mathbf{t}^{-1}, q^{-1})$ . Since  $(Y^\lambda)^\dagger = (Y'^\lambda)^{-1}$  for  $\lambda \in \Lambda$ , we obtain the following extension of [25, cor. 17] from theorem 4.8 and proposition 8.3.

**Corollary 8.5** (Bi-orthogonality relations). *For  $\lambda, \mu \in \Lambda$  with  $\lambda \neq \mu$  we have  $\langle E(\gamma_\lambda; \cdot), E'(\gamma_\mu^{-1}; \cdot) \rangle = 0$ .*

Recall that  $E^+(\gamma_\lambda; \cdot)$  ( $\lambda \in \Lambda^+$ ) can be characterized as the unique solution of the eigenvalue equation

$$Lf = (m_{\epsilon_1}(\gamma_\lambda) - m_{\epsilon_1}(\gamma_0))f, \quad f \in \mathcal{A}^W$$

which takes the value one at  $x_0^{\pm 1}$ , where  $L$  (5.1) is Koornwinder's second order  $q$ -difference operator. Since  $L$  and the eigenvalue  $m_{\epsilon_1}(\gamma_\lambda) - m_{\epsilon_1}(\gamma_0)$  are invariant under replacement of the parameters  $(\mathbf{t}, q)$  by their inverses  $(\mathbf{t}^{-1}, q^{-1})$ , we derive that

$$E^+(\gamma_\lambda; x; \mathbf{t}; q) = E^+(\gamma_\lambda^{-1}; x; \mathbf{t}^{-1}; q^{-1}), \quad \lambda \in \Lambda^+. \quad (8.12)$$

Combined with (8.7), corollary 8.5 and theorem 6.6, we re-obtain Koornwinder's [14] orthogonality relations for the symmetric Koornwinder polynomials:

**Corollary 8.6** (Orthogonality relations). *For  $\lambda, \mu \in \Lambda^+$  with  $\lambda \neq \mu$ , we have  $\langle E^+(\gamma_\lambda; \cdot), E^+(\gamma_\mu; \cdot) \rangle_+ = 0$ .*

We write  $\text{Spec}(Y') = \{\gamma_\lambda^{-1} \mid \lambda \in \Lambda\}$  for the spectrum of the  $Y'$ -operators, and  $F = F_{\mathbf{t}, q}$  for the linear space of functions  $g : \text{Spec}(Y') \rightarrow \mathbb{C}$  with finite support. We define a  $\mathcal{W}$ -module structure on  $F$  by

$$(wg)(\gamma_\lambda^{-1}) = g(\gamma_{w^{-1}\lambda}^{-1}), \quad g \in F, \quad w \in \mathcal{W}, \quad \lambda \in \Lambda.$$

**Definition 8.7.** *We call the linear map  $\mathcal{F} = \mathcal{F}_{\mathbf{t}, q} : \mathcal{A} \rightarrow F$  defined by*

$$\mathcal{F}(f)(\gamma) = \langle f, E'(\gamma; \cdot) \rangle, \quad f \in \mathcal{A}, \quad \gamma \in \text{Spec}(Y') \quad (8.13)$$

*the non-symmetric Koornwinder transform.*

Observe that  $\mathcal{F}$  is injective since  $\langle \cdot, \cdot \rangle$  is non-degenerate, and that  $\mathcal{F}$  is surjective by corollary 8.5.

In the next proposition we present an action of the double affine Hecke algebra  $\tilde{\mathcal{H}}$  on  $F$  in terms of spectral difference-reflection operators, and we relate it to the

action of  $\mathcal{H}$  on  $\mathcal{A}$  via the non-symmetric Koornwinder transform  $\mathcal{F}$  and the duality isomorphism  $\Phi$ .

**Proposition 8.8.** *The applications*

$$\begin{aligned} (\tilde{T}_i g)(\gamma) &= \tilde{t}_i g(\gamma) + \tilde{t}_i^{-1} \tilde{v}_{a_i}(\gamma) ((s_i g)(\gamma) - g(\gamma)), & i \in \{0, \dots, n\}, \\ (f(\tilde{z})g)(\gamma) &= f(\gamma)g(\gamma), & f \in \mathcal{A} \end{aligned} \quad (8.14)$$

where  $g \in F$  and  $\gamma \in \text{Spec}(Y')$ , uniquely extend to an action of  $\tilde{\mathcal{H}}$  on  $F$ . Furthermore,

$$\mathcal{F}(X(f)) = \Phi(X)\mathcal{F}(f), \quad X \in \mathcal{H}, f \in \mathcal{A}, \quad (8.15)$$

where  $\Phi$  is the duality isomorphism.

*Proof.* The intertwining property (8.15) can be proved by checking it for the algebraic generators  $U_n$ ,  $T_i$  and  $Y_i$  ( $i = 1, \dots, n$ ) of  $\mathcal{H}$  using proposition 8.3 and proposition 7.8. The fact that (8.14) defines an action of  $\tilde{\mathcal{H}}$  on  $F$  follows then from (8.15) since  $\Phi$  is an algebra isomorphism and  $\mathcal{F}$  is bijective.  $\square$

Next we determine the inverse of the non-symmetric Koornwinder transform  $\mathcal{F}$ . We let  $\mathcal{G} = \mathcal{G}_{\mathbf{t}, q} : F \rightarrow \mathcal{A}$  be the linear map defined by

$$(\mathcal{G}g)(x) = \sum_{\lambda \in \Lambda} g(\gamma_\lambda^{-1}) E(\gamma_\lambda; x; \mathbf{t}; q) w(\gamma_\lambda^{-1}; \tilde{\mathbf{t}}; q), \quad g \in F, \quad (8.16)$$

where the discrete weights  $\tilde{w}(\gamma_\lambda^{-1}) = w(\gamma_\lambda^{-1}; \tilde{\mathbf{t}}; q)$  ( $\lambda \in \Lambda$ ) are defined as follows:

$$w(\gamma_\lambda^{-1}; \tilde{\mathbf{t}}; q) = \mathcal{C}(\gamma_\lambda^{-1}; \tilde{\mathbf{t}}; q) w_+(\gamma_{\lambda^+}^{-1}; \tilde{\mathbf{t}}; q), \quad (8.17)$$

with  $\tilde{w}_+(\gamma_\mu^{-1}) = w_+(\gamma_\mu^{-1}; \tilde{\mathbf{t}}; q)$  for  $\mu \in \Lambda^+$  given by the multiple residue

$$w_+(\gamma_\mu^{-1}; \tilde{\mathbf{t}}; q) = \text{Res}_{x_1 = \gamma_\mu^{-\epsilon_1}} \left( \text{Res}_{x_2 = \gamma_\mu^{-\epsilon_2}} \left( \dots \text{Res}_{x_n = \gamma_\mu^{-\epsilon_n}} \left( \frac{\Delta_+(x; \tilde{\mathbf{t}}; q)}{x_1 \cdots x_n} \right) \dots \right) \right). \quad (8.18)$$

Using the second expression in (8.2) together with (4.4), it is easily verified that the discrete weights  $\tilde{w}(\gamma)$  and  $\tilde{w}_+(\gamma)$  ( $\gamma \in \text{Spec}(Y')$ ) are well defined and non-zero for generic parameters  $\mathbf{t}$  and  $q$  (in fact, all residues in (8.18) are taken at simple poles).

**Proposition 8.9.** *We have*

$$\mathcal{G}(\tilde{X}g) = \Phi^{-1}(\tilde{X})\mathcal{G}(g), \quad \tilde{X} \in \tilde{\mathcal{H}}, g \in F.$$

*Proof.* The proof for  $\tilde{X} = \tilde{z}^\lambda$  with  $\lambda \in \Lambda$  is immediate. Hence it suffices to check the intertwining property for  $\tilde{X} = \tilde{T}_i$  ( $i = 0, \dots, n$ ). Let  $g \in F$ . By proposition 7.8 we have for  $i = 0, \dots, n$ ,

$$\mathcal{G}(\tilde{T}_i g) - \Phi^{-1}(\tilde{T}_i)(\mathcal{G}g) = \tilde{t}_i^{-1} \sum_{\lambda \in \Lambda} h_i(\gamma_\lambda; \cdot) \tilde{v}_{a_i}(\gamma_\lambda^{-1}) \tilde{w}(\gamma_\lambda^{-1})$$

with  $h_i(\gamma_\lambda; \cdot) \in \mathcal{A}$  given by

$$h_i(\gamma_\lambda; \cdot) = g(\gamma_{s_i \cdot \lambda}^{-1}) E(\gamma_\lambda; \cdot) - g(\gamma_\lambda^{-1}) E(\gamma_{s_i \cdot \lambda}; \cdot).$$

Since  $h_i(\gamma_{s_i \cdot \lambda}; \cdot) = -h_i(\gamma_\lambda; \cdot)$  for  $i = 0, \dots, n$  and  $\lambda \in \Lambda$ , it thus suffices to prove that

$$\tilde{v}_{a_i}(\gamma_\lambda^{-1}) \tilde{w}(\gamma_\lambda^{-1}) = \tilde{w}_+(\gamma_{\lambda^+}^{-1}) \prod_{\alpha \in \Sigma^- \cup \{a_i\}} \tilde{v}_\alpha(\gamma_\lambda^{-1}) \quad (8.19)$$

is invariant under replacement of  $\lambda \in \Lambda$  by  $s_i \cdot \lambda$  for all  $i \in \{0, \dots, n\}$  and all  $\lambda \in \Lambda$ . For  $i \in \{1, \dots, n\}$  this is immediate by lemma 4.6.

As usual, the proof for the affine part of the statement (the case  $i = 0$ ) is more subtle. We begin by rewriting  $\tilde{w}(\gamma_\lambda^{-1})$  as a (kind of) multiple residue of  $\tilde{\Delta}(x)$  at  $x = \gamma_\lambda^{-1}$ , where  $\tilde{\Delta}(x) = \Delta(x; \tilde{\mathbf{t}}; q)$ . This can be done using the  $W$ -invariance of the weight function  $\Delta_+(\cdot; \tilde{\mathbf{t}}; q)$ . The result is as follows.

Let  $u_\lambda \in S_n$  be the component in  $S_n$  of the minimal coset representative  $w_\lambda \in W$  with respect to the semi-direct product structure  $W = S_n \ltimes (\pm 1)^n$ , and let  $n_\lambda = \#\{i \in \{1, \dots, n\} \mid \lambda_i < 0\}$ . Then we have

$$\tilde{w}(\gamma_\lambda^{-1}) = \mathbf{Res}_{x=\gamma_\lambda^{-1}} \left( \frac{\tilde{\Delta}(x)}{x_1 \cdots x_n} \right) \quad (8.20)$$

for all  $\lambda \in \Lambda$ , where the multiple residue at  $x = \gamma_\lambda^{-1}$  is defined by

$$\mathbf{Res}_{x=\gamma_\lambda^{-1}}(\cdot) = (-1)^{n_\lambda} \mathbf{Res}_{x_{u_\lambda(1)}=\gamma_\lambda^{-\epsilon_{u_\lambda(1)}}} \left( \mathbf{Res}_{x_{u_\lambda(2)}=\gamma_\lambda^{-\epsilon_{u_\lambda(2)}}} \left( \cdots \mathbf{Res}_{x_{u_\lambda(n)}=\gamma_\lambda^{-\epsilon_{u_\lambda(n)}}} \left( \cdot \right) \cdots \right) \right).$$

In particular, we obtain

$$\tilde{v}_{a_0}(\gamma_\lambda^{-1}) \tilde{w}(\gamma_\lambda^{-1}) = \mathbf{Res}_{x=\gamma_\lambda^{-1}} \left( \frac{\tilde{v}_{a_0}(x) \tilde{\Delta}(x)}{x_1 \cdots x_n} \right) \quad (8.21)$$

for all  $\lambda \in \Lambda$ . Now we consider (8.21) with  $\lambda$  replaced by  $s_0 \cdot \lambda$ . We first consider the changes in the multiple residue. By the proof of [24, thm. 5.3], we have  $w_{s_0 \cdot \lambda} = s_{\epsilon_1} w_\lambda$  for all  $\lambda \in \Lambda$ , i.e.  $n_{s_0 \cdot \lambda} = n_\lambda \pm 1$  and  $u_{s_0 \cdot \lambda} = u_\lambda$ . Secondly,

$$(\gamma_{s_0 \cdot \lambda}^{-1})^{\epsilon_i} = \begin{cases} q \gamma_\lambda^{\epsilon_1} & \text{if } i = 1, \\ \gamma_\lambda^{-\epsilon_i} & \text{if } i = 2, \dots, n \end{cases}$$

by lemma 4.6. So if we replace the residue at  $x_1 = \gamma_\lambda^{-\epsilon_1}$  by the residue at  $x_1 = q \gamma_\lambda^{\epsilon_1}$  in the definition of the multiple residue at  $x = \gamma_\lambda^{-1}$ , then we obtain minus the multiple residue at  $x = \gamma_{s_0 \cdot \lambda}^{-1}$ . On the other hand, we know by the proof of proposition 8.3 that  $v_{a_0}(x) \Delta(x)$  is invariant under the action of  $s_0$ . So the invariance of (8.21) under replacement of  $\lambda$  by  $s_0 \cdot \lambda$  follows from the simple observation that

$$\mathbf{Res}_{y=y_0} \left( \frac{h(y)}{y} \right) = - \mathbf{Res}_{y=qy_0^{-1}} \left( \frac{h(y)}{y} \right)$$

when  $h(y)$  is a function depending on a single variable  $y$ , having a simple pole at  $y = y_0$ , and satisfying the invariance condition  $h(qy^{-1}) = h(y)$ .  $\square$

**Theorem 8.10.** *We have  $\mathcal{G} \circ \mathcal{F} = k \text{Id}_{\mathcal{A}}$  and  $\mathcal{F} \circ \mathcal{G} = k \text{Id}_{\mathcal{F}}$  with  $k = k_{\mathbf{t},q} = w(\gamma_0^{-1}; \tilde{\mathbf{t}}; q) \langle 1, 1 \rangle_{\mathbf{t},q}$ . In particular, we have for  $\lambda, \mu \in \Lambda$ ,*

$$\frac{\langle E(\gamma_\lambda; \cdot), E'(\gamma_\mu^{-1}; \cdot) \rangle}{\langle 1, 1 \rangle} = \delta_{\lambda, \mu} \frac{\tilde{w}(\gamma_0^{-1})}{\tilde{w}(\gamma_\lambda^{-1})} \quad (8.22)$$

where  $\delta_{\lambda, \mu}$  is the Kronecker delta.

*Proof.* By proposition 8.8 and proposition 8.9 we have

$$\mathcal{G}(\mathcal{F}(f)) = \mathcal{G}(\mathcal{F}(f(z)1)) = f(z) \mathcal{G}(\mathcal{F}(1)), \quad \forall f \in \mathcal{A}. \quad (8.23)$$

Furthermore, it follows from corollary 8.5 that

$$\mathcal{G}(\mathcal{F}(E(\gamma; \cdot))) = \langle E(\gamma; \cdot), E'(\gamma^{-1}; \cdot) \rangle \tilde{w}(\gamma^{-1}) E(\gamma; \cdot) \quad (8.24)$$



for  $\gamma \in \text{Spec}(Y)$ . Formula (8.24) reduces to  $\mathcal{G}(\mathcal{F}(1)) = k1$  when  $\gamma = \gamma_0$ , with the constant  $k$  as given in the statement of the theorem. Combined with (8.23) it follows that  $\mathcal{G} \circ \mathcal{F} = k \text{Id}_{\mathcal{A}}$ . Since  $\mathcal{F}$  is bijective, we then also have  $\mathcal{F} \circ \mathcal{G} = k \text{Id}_F$ .

It remains to prove (8.22). By corollary 8.5, we only have to prove (8.22) when  $\mu = \lambda$ . We fix  $\gamma = \gamma_\lambda \in \text{Spec}(Y)$ ,  $\lambda \in \Lambda$ . Since  $\mathcal{G} \circ \mathcal{F} = k \text{Id}_{\mathcal{A}}$ , it follows that  $\mathcal{G}(\mathcal{F}(E(\gamma; \cdot))) = k E(\gamma; \cdot)$ . Comparing this outcome with the right-hand side of (8.24), we obtain

$$\langle E(\gamma; \cdot), E'(\gamma^{-1}; \cdot) \rangle \tilde{w}(\gamma^{-1}) = k = \langle 1, 1 \rangle \tilde{w}(\gamma_0^{-1})$$

which yields the desired result.  $\square$

**Corollary 8.11.** *For all  $\lambda, \mu \in \Lambda^+$ , we have*

$$\frac{\langle E^+(\gamma_\lambda; \cdot), E^+(\gamma_\mu; \cdot) \rangle_+}{\langle 1, 1 \rangle_+} = \delta_{\lambda, \mu} \frac{\tilde{w}_+(\gamma_0^{-1})}{\tilde{w}_+(\gamma_\lambda^{-1})}.$$

*Proof.* First of all, observe that  $E^+(\gamma_{\lambda+}; \cdot) = C_+ E(\gamma_\lambda; \cdot)$  for all  $\lambda \in \Lambda$  and that  $(C_+)^{\dagger} = C'_+$ , where  $C_+ \in \mathcal{H}$  (respectively  $C'_+ \in \mathcal{H}'$ ,  $\tilde{C}_+ \in \tilde{\mathcal{H}}$ ) is the idempotent corresponding to the trivial representation of the underlying finite Hecke algebra of type  $C_n$ . Combined with proposition 8.3, (8.7) and (8.12) we can rewrite  $\mathcal{F}_+ = \mathcal{F}|_{\mathcal{A}^W}$  as

$$(\mathcal{F}_+ f)(\gamma_\lambda^{-1}) = \frac{K}{|W|} \langle f, E^+(\gamma_{\lambda+}; \cdot) \rangle_+, \quad f \in \mathcal{A}^W, \lambda \in \Lambda.$$

In particular, the symmetric Koornwinder transform  $\mathcal{F}_+$  maps into

$$F^W = \{g \in F \mid wg = g \ \forall w \in W\} = \{g \in F \mid \tilde{C}_+ g = g\}.$$

Similarly, since  $\Phi(C_+) = \tilde{C}_+$ , we derive from proposition 8.9, (8.17) and lemma 8.2 that  $\mathcal{G}_+ = \mathcal{G}|_{F^W}$  can be rewritten as

$$(\mathcal{G}_+ g)(x) = \tilde{K} \sum_{\lambda \in \Lambda^+} g(\gamma_\lambda^{-1}) E^+(\gamma_\lambda; x) \tilde{w}_+(\gamma_\lambda^{-1}),$$

where  $\tilde{K} = K_{\tilde{\mathbf{t}}, q}$ . Using these alternative descriptions for  $\mathcal{F}_+$  and  $\mathcal{G}_+$  together with the orthogonality relations for the symmetric Koornwinder polynomials (see corollary 8.6), we obtain

$$\mathcal{G}_+(\mathcal{F}_+(E^+(\gamma_\lambda; \cdot))) = \frac{K \tilde{K}}{|W|} \langle E^+(\gamma_\lambda; \cdot), E^+(\gamma_\lambda; \cdot) \rangle_+ \tilde{w}_+(\gamma_\lambda^{-1}) E^+(\gamma_\lambda; \cdot)$$

for all  $\lambda \in \Lambda^+$ . On the other hand, by theorem 8.10, we have  $\mathcal{G}_+(\mathcal{F}_+(E^+(\gamma_\lambda; \cdot))) = k E^+(\gamma_\lambda; \cdot)$  for  $\lambda \in \Lambda^+$  with

$$k = \langle 1, 1 \rangle \tilde{w}(\gamma_0^{-1}) = \frac{K \tilde{K}}{|W|} \langle 1, 1 \rangle_+ \tilde{w}_+(\gamma_0^{-1}).$$

Comparing the two different outcomes for  $\mathcal{G}_+(\mathcal{F}_+(E^+(\gamma_\lambda; \cdot)))$ , we obtain the desired result in case  $\mu = \lambda$ . The off-diagonal case is covered by the orthogonality relations for the symmetric Koornwinder polynomials, see corollary 8.6.  $\square$

*Remark 8.12.* The discrete weights  $\tilde{w}_+(\gamma_\lambda^{-1}) = w_+(\gamma_\lambda^{-1}; \tilde{\mathbf{t}}; q)$  ( $\lambda \in \Lambda^+$ ) also appear as weights in a partly discrete orthogonality measure for the symmetric Koornwinder polynomials, see [26]. In particular, using the expression [26, prop. 4.1] for the discrete weights, it can be shown that the ratio  $\langle E^+(\gamma_\lambda; \cdot), E^+(\gamma_\lambda; \cdot) \rangle_+ / \langle 1, 1 \rangle_+ = \tilde{w}_+(\gamma_0^{-1}) / \tilde{w}_+(\gamma_\lambda^{-1})$  is equal to

$$\prod_{i=1}^n \left\{ \frac{(q^{-1}abcdt^{4(n-i)}; q)_{2\lambda_i} (c^2t^{4(n-i)})^{\lambda_i}}{(abcdt^{4(n-i)}; q)_{2\lambda_i}} \cdot \frac{(qt^{2(n-i)}, abt^{2(n-i)}, adt^{2(n-i)}, bdt^{2(n-i)}; q)_{\lambda_i}}{(act^{2(n-i)}, bct^{2(n-i)}, cdt^{2(n-i)}, q^{-1}abcdt^{2(n-i)}; q)_{\lambda_i}} \right\} \\ \cdot \prod_{1 \leq i < j \leq n} \frac{(abcdt^{2(2n-i-j-1)}, q^{-1}abcdt^{2(2n-i-j)}; q)_{\lambda_i + \lambda_j}}{(abcdt^{2(2n-i-j)}, q^{-1}abcdt^{2(2n-i-j+1)}; q)_{\lambda_i + \lambda_j}} \frac{(qt^{2(j-i-1)}, t^{2(j-i)}; q)_{\lambda_i - \lambda_j}}{(qt^{2(j-i)}, t^{2(j-i+1)}; q)_{\lambda_i - \lambda_j}}$$

for all  $\lambda \in \Lambda^+$ , where  $(y_1, \dots, y_m; q)_k = \prod_{j=1}^m (y_j; q)_k$  and  $(y; q)_k = \prod_{j=0}^{k-1} (1 - yq^j)$  for  $k \in \mathbb{Z}_+$ .

## 9. EVALUATION FORMULAS

In this section we give an explicit expression for the value of the non-symmetric Koornwinder polynomial  $P_\lambda(\cdot) = P_\lambda(\cdot; \mathbf{t}; q)$  at  $x_0^{-1}$ . We start with two preliminary lemmas.

**Lemma 9.1.** *For  $\lambda \in \Lambda$ , we have*

$$\frac{\langle P_\lambda, E'(\gamma_\lambda^{-1}; \cdot) \rangle}{\langle P_{\lambda^+}, E'(\gamma_{\lambda^+}^{-1}; \cdot) \rangle} = \tilde{t}_{w_\lambda}^2 \prod_{\alpha \in \Sigma^+ \cap w_\lambda^{-1} \Sigma^-} \frac{1}{\tilde{v}_\alpha(\gamma_{\lambda^+}^{-1})}.$$

*Proof.* We prove the lemma using the non-affine intertwiners  $S_i$  ( $i = 1, \dots, n$ ). For the moment, we fix  $\lambda \in \Lambda$  and  $i \in \{1, \dots, n\}$  such that  $\mu = s_i \lambda \neq \lambda$ . We first give some additional properties of  $S_i$  which we will need for the proof.

The intertwiner  $S_i$  is self-adjoint, i.e.  $(S_i)^\dagger = S'_i$ , where  $S'_i = [T'_i, (Y')^{a_i}] \in \mathcal{H}'$ . This can be checked most easily in the image of the duality anti-isomorphism  $\Psi_{\mathbf{t}^{-1}, q^{-1}}$ . It follows from proposition 7.8(ii) that  $S'_i E'(\gamma_\lambda^{-1}; \cdot) = L_{i, \lambda} E'(\gamma_\mu^{-1}; \cdot)$  with  $L_{i, \lambda} = (\gamma_\lambda^{-a_i} - \gamma_\lambda^{a_i}) \tilde{t}_i^{-1} \tilde{v}_{a_i}(\gamma_\lambda^{-1})$ . On the other hand,  $S_i P_\lambda = K_{i, \lambda} P_\mu$  with  $K_{i, \lambda} = (\gamma_\lambda^{a_i} - \gamma_\lambda^{-a_i}) \eta_i(\gamma_\lambda)$  by corollary 6.2. Combined with proposition 8.3 we obtain

$$\langle P_\mu, E'(\gamma_\mu^{-1}; \cdot) \rangle = \frac{1}{L_{i, \lambda}} \langle S_i P_\mu, E'(\gamma_\lambda^{-1}; \cdot) \rangle = \frac{K_{i, \mu}}{L_{i, \lambda}} \langle P_\lambda, E'(\gamma_\lambda^{-1}; \cdot) \rangle.$$

In particular, if  $\langle \lambda, a_i \rangle < 0$ , then

$$\langle P_\lambda, E'(\gamma_\lambda^{-1}; \cdot) \rangle = \frac{\tilde{t}_i^2}{\tilde{v}_{a_i}(\gamma_\mu^{-1})} \langle P_\mu, E'(\gamma_\mu^{-1}; \cdot) \rangle.$$

The ratio  $\langle P_\lambda, E'(\gamma_\lambda^{-1}; \cdot) \rangle / \langle P_{\lambda^+}, E'(\gamma_{\lambda^+}^{-1}; \cdot) \rangle$  can now be evaluated inductively using similar techniques as in the proof of theorem 6.6. This gives the desired result.  $\square$

Let  $\delta_{\gamma_\lambda^{-1}} \in F$  for  $\lambda \in \Lambda$  be the function which is equal to one at  $\gamma_\lambda^{-1}$  and zero otherwise.

**Lemma 9.2.** *For  $\lambda \in \Lambda^+$  we have*

$$(\Phi(z^\lambda)\delta_{\gamma_0^{-1}})(\gamma_\lambda^{-1}) = \tilde{t}_{\tau(-\lambda)}^{-1} \prod_{\beta \in R^+ \cap \tau(\lambda)R^-} \tilde{v}_\beta(\gamma_\lambda^{-1}).$$

*Proof.* Let  $\lambda \in \Lambda^+$ . It follows from (2.2) that  $\tau(-\lambda)$  is the unique element of minimal length in  $\mathcal{W}$  which maps  $\lambda$  to 0  $\in \Lambda$  under the dot-action. In particular, any element  $w \in \mathcal{W}$  which is smaller than  $\tau(-\lambda)$  with respect to the Bruhat order, maps  $\lambda$  to a non-zero element in  $\Lambda$ .

Since  $\lambda \in \Lambda^+$ , we have  $Y^\lambda = T_{\tau(\lambda)}$ , hence

$$\Phi(z^\lambda) = (\epsilon(z^\lambda))^{\tilde{t}'} = (\tilde{T}'_{\tau(\lambda)})^{\tilde{t}'} = \tilde{T}_{\tau(-\lambda)}^{-1}.$$

Let now  $\tau(-\lambda) = s_{i_1}s_{i_2}\cdots s_{i_r}$  be a reduced expression of  $\tau(-\lambda)$  in  $\mathcal{W}$ , then we obtain from proposition 8.8 and from the previous paragraph that

$$(\Phi(z^\lambda)\delta_{\gamma_0^{-1}})(\gamma_\lambda^{-1}) = \prod_{m=1}^r \tilde{t}_{a_{i_m}}^{-1} \tilde{v}_{a_{i_m}}(\gamma_{(s_{i_{m+1}}\cdots s_{i_r})\lambda}^{-1}),$$

where  $(s_{i_{m+1}}\cdots s_{i_r})\lambda$  should be read as  $\lambda$  when  $m = r$ . The lemma follows now easily from lemma 4.6 and remark 6.5.  $\square$

**Theorem 9.3.** *Let  $\lambda \in \Lambda$ , then*

$$P_\lambda(x_0^{-1}) = \frac{\tilde{w}(\gamma_\lambda^{-1})}{\tilde{w}(\gamma_0^{-1})} \tilde{t}_{w_\lambda}^2 \tilde{t}_{\tau(-\lambda^+)}^{-1} \prod_{\beta} \tilde{v}_\beta(\gamma_{\lambda^+}^{-1}),$$

where the product is taken over  $\beta \in (R^+ \cap \tau(\lambda^+)R^-) \setminus (\Sigma^+ \cap w_\lambda^{-1}\Sigma^-)$ .

*Proof.* First of all, observe that

$$\begin{aligned} P_\lambda(x_0^{-1})\langle E(\gamma_\lambda; \cdot), E'(\gamma_\lambda^{-1}; \cdot) \rangle &= \langle P_\lambda, E'(\gamma_\lambda^{-1}; \cdot) \rangle, \\ \langle P_\lambda, E'(\gamma_\lambda^{-1}; \cdot) \rangle &= \langle x^\lambda, E'(\gamma_\lambda^{-1}; \cdot) \rangle = \langle 1, 1 \rangle (\Phi(z^\lambda)\delta_{\gamma_0^{-1}})(\gamma_\lambda^{-1}) \end{aligned} \quad (9.1)$$

for all  $\lambda \in \Lambda$ . Indeed, the first formula is trivial, while the second formula is a direct consequence of corollary 8.5, the definition of the non-symmetric Koornwinder transform  $\mathcal{F}$  and its intertwining properties given in proposition 8.8.

We use now successively the first formula of (9.1), then lemma 9.1 and finally the second formula of (9.1) to arrive at

$$P_\lambda(x_0^{-1}) = \frac{\tilde{t}_{w_\lambda}^2 \langle 1, 1 \rangle}{\langle E(\gamma_\lambda; \cdot), E'(\gamma_\lambda^{-1}; \cdot) \rangle} (\Phi(z^{\lambda^+})\delta_{\gamma_0^{-1}})(\gamma_{\lambda^+}^{-1}) \prod_{\alpha \in \Sigma^+ \cap w_\lambda^{-1}\Sigma^-} \frac{1}{\tilde{v}_\alpha(\gamma_{\lambda^+}^{-1})}.$$

The theorem follows now from theorem 8.10 and lemma 9.2, combined with the inclusion  $\Sigma^+ \cap w_\lambda^{-1}\Sigma^- \subset R^+ \cap \tau(\lambda^+)R^-$ . This inclusion is a direct consequence of the inequality  $\langle \lambda^+, \alpha \rangle > 0$  for all  $\alpha \in \Sigma^+ \cap w_\lambda^{-1}\Sigma^-$ , see (6.15).  $\square$

**Corollary 9.4.**

$$P_\lambda^+(x_0^{\pm 1}) = \frac{\tilde{w}_+(\gamma_\lambda^{-1})}{\tilde{w}_+(\gamma_0^{-1})} \tilde{t}_{\tau(-\lambda)}^{-1} \prod_{\beta \in R^+ \cap \tau(\lambda)R^-} \tilde{v}_\beta(\gamma_\lambda^{-1}), \quad \lambda \in \Lambda^+. \quad (9.2)$$

*Proof.* This follows from the evaluation of  $P_\mu(x_0^{-1})$  ( $\mu \in \Lambda$ ) (see theorem 9.3), theorem 6.6(iii), (8.17) and lemma 8.2.  $\square$

*Remark 9.5.* It is straightforward to explicitly write down the roots  $R^+ \cap \tau(\lambda)R^-$  for  $\lambda \in \Lambda^+$ . Together with (4.4), (3.7), remark 8.12 and the  $\mathcal{W}$ -orbit structure of  $R$ , one can reformulate (9.2) now as

$$P_\lambda^+(x_0) = \prod_{i=1}^n \frac{(act^{2(n-i)}, bct^{2(n-i)}, cdt^{2(n-i)}, q^{-1}abcdt^{2(n-i)}; q)_{\lambda_i}}{(q^{-1}abcdt^{4(n-i)}; q)_{2\lambda_i} (ct^{2(n-i)})^{\lambda_i}} \cdot \prod_{1 \leq i < j \leq n} \frac{(q^{-1}abcdt^{2(2n-i-j+1)}; q)_{\lambda_i + \lambda_j} (t^{2(j-i+1)}; q)_{\lambda_i - \lambda_j}}{(q^{-1}abcdt^{2(2n-i-j)}; q)_{\lambda_i + \lambda_j} (t^{2(j-i)}; q)_{\lambda_i - \lambda_j}}$$

for  $\lambda \in \Lambda^+$ .

*Remark 9.6.* Van Diejen [9, thm. 5.1] proved the evaluation formula (9.2) for a five parameter sub-family of the symmetric Koornwinder polynomials. This result was indirectly extended to the complete six parameter family of symmetric Koornwinder polynomials by Sahi's [24] duality results.

## 10. THE GENERALIZED WEYL CHARACTER FORMULA AND THE CONSTANT TERM

In this section we discuss several results involving the anti-symmetric Koornwinder polynomials  $P_\lambda^-(\cdot) = P_\lambda^-(\cdot; \mathbf{t}; q)$  ( $\lambda \in \Lambda^{++}$ ) in an essential way. The most crucial result is the analogue of the Weyl character formula for the Koornwinder polynomials.

In order to state the generalized Weyl character formula, we need to introduce some notations first. Let  $\mathbf{q} = \{q_\beta\}_{\beta \in S}$  be the multiplicity function satisfying  $q_{a_0} = q_{a_0^\vee} = q_{a_n^\vee} = 1$ ,  $q_{a_i} = q^{1/2}$  ( $i \in \{1, \dots, n-1\}$ ) and  $q_{a_n} = q$ . For two multiplicity functions  $\mathbf{t}$  and  $\mathbf{t}'$ , we write  $\mathbf{t}\mathbf{t}'$  for the multiplicity function which takes the value  $t_\beta t'_\beta$  at  $\beta \in S$ . Then the generalized Weyl character formula is given by

$$P_{\lambda+\kappa}^-(x; \mathbf{t}; q) = \chi(x; \mathbf{t}; q) P_\lambda^+(x; \mathbf{t}\mathbf{t}; q), \quad \lambda \in \Lambda^+ \quad (10.1)$$

with  $\kappa$  given by (6.9) and with  $\chi(\cdot; \mathbf{t}; q) \in \mathcal{A}$  given by

$$\chi(x; \mathbf{t}; q) = x^\kappa \prod_{\alpha \in \Sigma^-} (1 - x^\alpha) v_\alpha(x; \mathbf{t}^{-1}; q^{-1}).$$

The proof of (10.1) is similar to the proof of the generalized Weyl character formula for Macdonald polynomials, see e.g. [18, 7.3].

The generalized Weyl character formula (10.1), together with the results of section 8, readily implies the norm relations

$$\begin{aligned} \frac{\langle P_\lambda^+(\cdot; \mathbf{t}; q), P_\lambda^+(\cdot; \mathbf{t}; q) \rangle_{\mathbf{t}, q}}{\langle P_\lambda^-(\cdot; \mathbf{t}; q), P_\lambda^-(\cdot; \mathbf{t}^{-1}; q^{-1}) \rangle_{\mathbf{t}, q}} &= t_\sigma^2 \frac{\langle P_\lambda^+(\cdot; \mathbf{t}; q), P_\lambda^+(\cdot; \mathbf{t}; q) \rangle_{+, \mathbf{t}, q}}{\langle P_{\lambda-\kappa}^+(\cdot; \mathbf{t}\mathbf{t}; q), P_{\lambda-\kappa}^+(\cdot; \mathbf{t}\mathbf{t}; q) \rangle_{+, \mathbf{t}\mathbf{t}, q}} \\ &= \prod_{\alpha \in \Sigma^+} \frac{v_\alpha(\gamma_\lambda^{-1}; \tilde{\mathbf{t}}; q)}{v_\alpha(\gamma_\lambda; \mathbf{t}; q)} \end{aligned} \quad (10.2)$$

for  $\lambda \in \Lambda^{++}$ . The norm relations (10.2) give an explicit description of the diagonal terms  $\langle P_\lambda^-(\cdot; \mathbf{t}; q), P_\lambda^-(\cdot; \mathbf{t}^{-1}; q^{-1}) \rangle_{\mathbf{t}, q}$  ( $\lambda \in \Lambda^{++}$ ) corresponding to the bi-orthogonality relations

$$\langle P_\lambda^-(\cdot; \mathbf{t}; q), P_\mu^-(\cdot; \mathbf{t}^{-1}; q^{-1}) \rangle_{\mathbf{t}, q} = 0 \quad \lambda, \mu \in \Lambda^{++} : \lambda \neq \mu \quad (10.3)$$

in terms of the quadratic norms of the symmetric Koornwinder polynomials. Furthermore, the norm relations (10.2) enables one to obtain a new proof for Gustafson's [11] evaluation of the constant term  $\langle 1, 1 \rangle_+$ . In fact, by (10.2), one can express  $\langle 1, 1 \rangle_{+, \mathbf{q}^m \mathbf{v}, q}$  in terms of

$$\langle P_{m\kappa}^+(\cdot; \mathbf{v}; q), P_{m\kappa}^+(\cdot; \mathbf{v}; q) \rangle_{+, \mathbf{v}, q} \quad (10.4)$$

for all positive integers  $m$ . Let now  $\mathbf{v}$  be a multiplicity function with  $v_\beta = 1$  for all  $\beta \in S$  of length two, then the corresponding orthogonality measure reduces to the (coordinate-wise) product measure of the one-variable Askey-Wilson polynomials. In particular, (10.4) can be expressed in terms of the quadratic norms of the Askey-Wilson polynomials, which were evaluated in [1] (see [22] for an affine Hecke algebraic approach). This yields an evaluation of (10.4), and hence of  $\langle 1, 1 \rangle_{+, \mathbf{q}^m \mathbf{v}, q}$ . By analytic continuation, we arrive at Gustafson's [11] result that the constant term  $|W|^{-1} \langle 1, 1 \rangle_{+, t, q}$  is equal to

$$\prod_{j=1}^n \frac{(t^2, t^{2(2n-j-1)}abcd; q)_\infty}{(q, t^{2(n-j+1)}, t^{2(n-j)}ab, t^{2(n-j)}ac, t^{2(n-j)}ad, t^{2(n-j)}bc, t^{2(n-j)}bd, t^{2(n-j)}cd; q)_\infty}.$$

*Remark 10.1.* In view of theorem 8.10, corollary 8.11, theorem 9.3, corollary 9.4 and Gustafson's constant term evaluation, we have arrived now at the stage that the quadratic norms (respectively diagonal terms) of the (non-)symmetric Koornwinder polynomials are completely explicit. In particular, the explicit evaluation of  $\langle P_\lambda^+, P_\lambda^+ \rangle_+$  ( $\lambda \in \Lambda^+$ ) which we thus obtain, can be seen to coincide with van Diejen's [9, thm. 5.2] explicit expression for  $\langle P_\lambda^+, P_\lambda^+ \rangle_+$ .

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